Computing the Expansion History of the Universe

J. Christiansen

Department of Physics

California Polytechnic State University

San Luis Obispo, CA

25 September 2013

Have you ever wondered how scientists determine the size or age of the universe? These bulk properties are a bit mysterious unless you can calculate them for yourself. The physical model of the expanding universe is the primary focus of our class and you'll be studying it in detail. This model relies on conservation of energy (kinetic and potential) and the thermodynamic properties of fluids and it is calculated in a coordinate system that expands with a scale factor, *a(t)*. To tie the model to reality, we have to understand how it impacts observable quantities like the redshift of a galaxy and the brightness or size of a galaxy. After all, it is only when we compare observations to a particular model that we find out whether the model is true or not. Models without supporting data are just fantasies of a creative mind.

In the first part of this project, we will use *a(t)* to compute observable quantities. In the second part, we will use the general solution to the Friedman Equation to determine *a(t)* for any universe of our choosing. This allows us to calculate the age and size of the universe. Then we'll explore the parameter space to determine how close Ryden's Benchmark model comes to the current best-measured parameters.

Part 1 – Observable Quantities

Telescopes generally point at astronomical sources to measure their photon intensity, spectra, and angular extent on the sky. At large distances these observables depend on the geometry and expansion rate of the universe. In fact, the expansion leads directly to an observed reddening of distant objects. We define this redshift in terms of the wavelength of light. If a distant galaxy emits light of a wavelength, λ_e, (*e* is for emitted) we will observe its redshifted wavelength, λ_o, (*o* is for observed) and the redshift is defined as (Ryden Eqn 2.4) [1]

$$z = (\lambda_o - \lambda_e)/\lambda_e \qquad (1)$$

The spectral lines of hydrogen, helium, and a number of other elements are routinely measured in undergraduate laboratories and you've probably seen this yourself when you studied optics. Remember that the each wavelength of light is a specific color. When the color changes, so does the wavelength. In astronomy, a spectrometer is used to measure the observed wavelength of astronomical objects with strong spectral lines. Since we already know the emitted wavelengths from our laboratory studies we can determine how the wavelength has changed and we call the relative change the redshift. A redshift can result from a Doppler shift due to the velocity of the astronomical source or from the expansion of the universe. In general, redshifts are a combination of the two. Beyond a redshift of about 0.03, however, the expansion of the universe dominates and the Doppler shift can be neglected.

To understand redshifts due to the expanding universe we need to see how length is defined during the expansion. The Robertson-Walker metric expresses the observed length, *ds*, in terms of the general relativistic space-time elements in spherical coordinates, *dt*, *dr*, and $d\Omega = \sqrt{d\theta^2 + sin^2\theta d\phi^2}$. (Ryden 3.25)

$$ds^2 = -c^2 dt^2 + a(t)^2 [dr^2 + S_K(r)^2 d\Omega^2] \qquad (2)$$

where *c* is the speed of light, *a(t)* is a unitless scale factor that describes the spatial expansion of the metric, and $S_k(r)$ accounts for the curvature of space.

$$S_k(r) = \begin{cases} R_0 \sin(r/R_0) & \text{for } \kappa = +1 \\ r & \text{for } \kappa = 0 \\ R_0 \sinh(r/R_0) & \text{for } \kappa = -1 \end{cases} \qquad (3)$$

where R_0 is the radius of curvature of the metric. Our universe appears to be flat with $S_k(r) = r$, but the metric allows for positive curvature, κ=+1, and negative curvature, κ=-1. Notice that the flat metric reduces to spherical coordinates with the additional special relativistic term, *-c dt*, and the expansion scale factor, *a(t)*. For convenience we set the scale factor to unity at the present time, $a(t_o) = 1$.

The curvature, R_0, and the sign of the curvature, κ, are determined from the Friedman equation: (Ryden 4.31)

$$\frac{\kappa}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1) \qquad (4)$$

The curvature, R_0 is related to whether the total energy density is greater or less than the critical energy density ($\Omega_0 = \Omega_{m,0}+\Omega_{r,o}+\Omega_{\Lambda,0}$). This is one of the few times that we can determine two variables with one equation. For example, if Ω_0>1, then the right-hand side is positive, κ=+1, and with this information we can solve for the radius of curvature, $R_0 = \frac{c}{H_0\sqrt{\Omega_0 - 1}}$.

Locations at fixed coordinates, *r*, θ, and ϕ in this metric are called comoving because they are observed to move in relation to each other by the scale factor, *a(t)*. We use light traveling between a comoving emission and observation point to measure the comoving distance interval, *dr*. Let's set the origin of the coordinate system at the telescope that observes the light. In this coordinate system a photon travels radially toward the observation point at a constant angle, θ and ϕ, from the emitting source.

This means that $d\Omega=0$. Furthermore, light travels along null geodesics defined by $ds = 0$ which allows us to solve for the comoving distance interval, $dr = c\, dt/a(t)$.

Two important results come from this. First, if we consider the wavelength of a photon $\lambda = cdt$ where dt is the period of the photon's oscillation, we find that this is $a(t_e)dr$ at the emission time and $a(t_o)dr$ when it is observed. For these comoving observers, dr is the same at both times and it is easy to show that the scale factor of the expanding universe, $a(t)$, is related to the redshift by: (Ryden Eqn 3.46)

$$z = \frac{1}{a(t_e)} - 1 \tag{5}$$

where we have set today's scale factor to unity, $a(t_o) = 1$. By measuring the redshift, z, of an astronomical source as a shift in wavelength, we learn the value of the scale factor at the time the light was emitted.

Second, we find the line-of-sight distance to the source at the time we observe it. This is called the conformal distance and it is found by integrating over time from the observed time, t_o, backward to the emission time, t_e. (Ryden Eqns. 3.39 and 5.35)

$$D_c = \int_0^{D_c} dr = c \int_{t_e}^{t_o} \frac{dt}{a(t)} \tag{6}$$

You may remember the importance of the proper distance in relativity. It is defined as the smallest distance observed by all commoving observers at a single time, t_i. Eqn. 2 shows us that smallest distance occurs when time is a constant, $dt = 0$ and the radial proper distance is $D_p(t_i) = \int ds = a(t_i) \int dr$. The proper distance today is the conformal distance $D_p(t_o)=D_c$. In an expanding universe the proper distance was smaller at the time the light was emitted $D_p(t_e)=a(t_e)D_c$.

Now that we have good definitions for distance, we can talk about what people see in their telescopes. The distance factors are derived in Ryden Ch. 7 for the flat universe where $S_k(D_C)=D_C$. Here we extend the discussion to include curvature. The observed width of a galaxy on the sky, $\Delta\Omega$, is related to the galaxy's diameter:

$$\text{true galaxy diameter} = D_A\, \Delta\Omega \tag{7}$$

where D_A is the angular diameter distance. To get a feel for this, imagine that one night you look up and see the moon. If you extend your arm to point at one edge of the moon and then move your arm to point at the other edge of the moon, the angle that your arm moves is $\Delta\Omega$. In highschool, when you took geometry, the galaxy diameter was called the arc length and you may recognize that Eqn. 7 in flat polar coordinates becomes $ds = rd\theta$. A close look at the metric in Eqn. 2 shows that D_A is simply defined by the coefficients of the $d\Omega$ term:

$$D_A = a(t_e) S_K(D_C). \tag{8}$$

Similarly, the observed brightness of a source depends on how far away it is. Imagine that you are looking at headlights in the distance on a dark night. As the headlights get closer to you, you perceive them as brighter. The headlights don't change their luminosity, rather your observation of them changes. The observed brightness of a source is characterized by the flux of photons into the aperture of a telescope during the exposure time and has units of photons/(m^2 s). Since photons are emitted in all directions, the fraction that make it into a fixed aperture at a distance, r, goes as the inverse of a spherical surface, $1/4\pi r^2$. The brightness also depends on the intrinsic luminosity of the source, defined by the total number of emitted photons/second in all directions. The observed flux is related to the luminosity by the luminosity distance, D_L.

$$\text{measured flux} = \text{true luminosity}/(4\pi D_L^2) \tag{9}$$

The luminosity distance is constructed to make the equation look geometrical, but since the photons spread out over a spherical area related to $d\Omega$ and are also redshifted during transit, it depends both on the curvature of the universe and the redshift.

$$D_L = S_K(D_C)/a(t_e) \tag{10}$$

Finally, it is rare to measure a luminosity or flux directly. Astronomers usually work with the logarithm of the flux and describe the brightness of a source by its magnitude. The apparent magnitude, m, of a source is given by: (Ryden Eqn. 7.48)

$$m = M + 5\log(\frac{D_L}{10pc}) \tag{11}$$

where M is the absolute magnitude and the second term is the distance modulus, $DM = 5\log(D_L/10pc)$. Notice that the distance modulus depends only on the luminosity distance which can be computed directly from the metric at any emission time. A prediction of DM exists for every specific cosmological model of $S_k(r)$ and $a(t_e)$. Direct tests of the expansion have been made by measuring the apparent magnitude for sources with known absolute magnitude and comparing the difference, $m - M$, to the predicted distance modulus, DM. These tests lead to the 2011 Nobel Prize in Physics awarded to Saul Perlmutter, Brian Schmidt, and Adam Riess [2].

Key point: All of these measurable quantities can be computed if we can just figure out R_0, H_0, and $a(t)$: two constants and a function.

There are a few cases where the scale factor can be computed analytically and in this part of the project it's good to start with one of those. The solution for the Matter Only universe ($\Omega_0 = \Omega_{m,0} = 1$) is

$$H_0(t_e - t_o) = \frac{2}{3}(a^{\frac{3}{2}} - 1) \tag{12}$$

where H_0 is the Hubble constant. Note: H_0 is not a function of t_e-t_o, but rather, the left-hand side is $H_0 \times (t_e - t_o)$. The function is plotted as a dotted line in Ryden's Figure 6.1. We will choose t_o=now, and measure t_e, as a time in the past or future. Right now, $t_e = t_o$, and the left-hand side is zero. What is a(now) so that the right-hand side is also zero? Let's use $a(t)$ to understand redshift, and the distance factors.

General Instructions: If you wish to do this assignment without the step-by-step instructions, feel free to pick any computing language of your choice. Start by defining the parameters $H_0 = 70$ km/s/Mpc and $\Omega_0 = 1$, the constant, *c,* and conversions. Since we want to get about 4 significant figures of accuracy out of this computation, we need to use constants and conversions that are accurate to 6 significant figures as shown in Table 1 below. Next, make an array of log(a) from -6 to 0.5, incrementing in steps of 0.01 or so. Make additional arrays from the first to hold the values of *a, z,* $H_0(t_e\text{-}t_0)$. Debug the results using columns A-F in the spreadsheet shown in Figure 3 and by reproducing Figure 1. What is the age of the universe?

Next, integrate Eqn. 6 to find D_c being very careful to set the integration limits from t_0 to t_e. You may use a trapezoid method, Simpson's method, or Romberg's method. Set $D_c = 0$ at t_0 and then integrate backward to an emission time, t_e, in the past. Again, check that D_c is correct using the spreadsheet in Figure 3. Next, find κ and R_0 so that you can compute $S_k(D_c)$. To check positive curvature: set Ω_0= 1.05 and check that S_k = 8275.94 Mpc where log(a) = -6. Then set Ω_0=0.95 and check that S_k = 8845.48 Mpc where log(a) = -6. Skip ahead to page 8 where it says **Report**.

Excel Instructions: If you prefer more explanation and a detailed approach, here's how to do the computation in Excel.

A) Our first objective is to compute a(t). One way to do this is to use Eqn. 12 to find an expression for $a(t_e)$. In later calculations, it won't be possible to do this, so we want to get good at using Eqn.12 as is. The time scales of interest to us extend from seconds to billions of years. To cover all the time scales of interest, start with log(a) instead of a. Take a look at the example spreadsheet in Figure 3 to see how the log(a) column should look. Create a spreadsheet column log(a). Compute a from the log(a) and then use Eqn. 12 to compute $H_0(t_e - t_0)$ from a. This should produce columns A, B, and C in your spreadsheet. Check them by recreating the line in Figure 1 for yourself.

B) Column D in the spreadsheet shown in Figure 3 is emission time, t_e. It is easily computed using H_0 = 70 km/s/Mpc and the time right now, t_o = 0 seconds. For parameters, like H_0, you'll want to put them in a cell at the top and use them in equations. If you type them in all over the place, you'll have to debug the code every time you change their value. **If you don't know how to anchor a number in an equation, please get help.** It's something every college student should learn to do. **Notice that the spreadsheet is color-coded**. Blue cells are computed with equations (they shouldn't have any numbers typed in by hand). Black cells are cells you have to enter by hand.

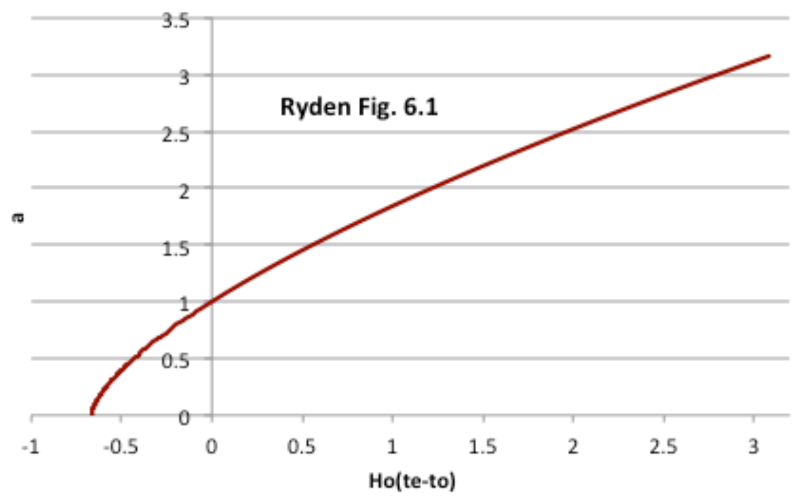


Figure 1: Recreation of the dotted line in the lower panel of Ryden Fig. 6.1 for the Matter Only universe.

C) Column E is the age of the universe in years. Notice that the emission time is measured backward from now and that age is measured forward from the Big Bang. If we start the age at 0 years, what is the age today?

D) Compute the redshift column using Eqn. 5.

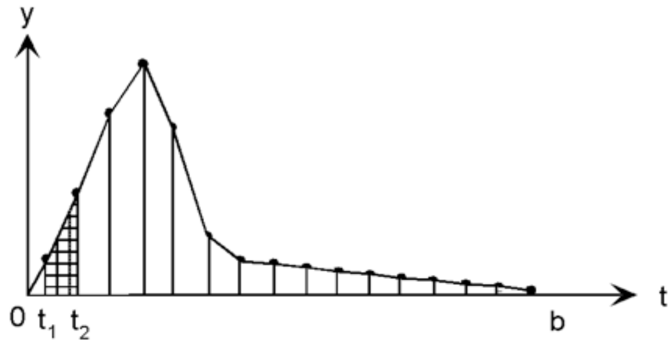
E) We want to compute some more parameters and constants before tackling the distance factors. It's important to use constants and conversion factors that are accurate to 6 significant figures. See Table 1.

Go back and **fix the conversions that you used in part C**. They need to be accurate. You'll also need to calculate the Hubble Time, $t_H = 1/H_0$, and the Hubble Distance, $D_H=c/H_0$. So find a spot at the top to pre-compute them. Check that you are using the constants and conversions correctly using the color code. Blue cells should have equations that refer to other cells. The black cells have all the input information needed.

speed of light	2.99792×10^5 km/s
Seconds/year (including leap seconds)	3.15581×10^7
Mpc/km	3.24078×10^{-20}

Table 1: Parameters and conversions with 6 significant figures.

F) The distance factors involve the integral shown in Eqn. 6. To compute the integral numerically, we'll find the area under the function, *f(t)=c/a(t)* where *f(t)* is plotted on the *y*-axis. A schematic of this is shown in Figure 2.



The area of the shaded trapezoid above is

$$Area = (t_2 - t_1)\left[\frac{f(t_1) + f(t_2)}{2}\right]$$

Figure 2: Schematic used to describe numerical integration. Note that the y-axis of the plot is f(t).

Column G in the spreadsheet below is the area of each trapezoid formed by two *y* values and two *x* values in a column. The time difference comes from Column D and the scale factors come from Column B. Compute column G.

To integrate Eqn. 6, we need to add up the trapezoidal areas in column G between our integration limits. This is where it gets tricky because we don't want to start at the beginning of the universe, but rather at the lower integration limit, which is the current time, t_o. We want D_c, the distance to be zero at the current time: **put a zero in the column H cell where t_e = 0**. This means that light emitted right now is at zero distance from us. Now we'll add the trapezoids above to find the distance travelled by light emitted in the past. It's simplest if you have an equation like H9 = H10+G9 in your spreadsheet. Make sure column H is correct.

G) Next we need to tackle the curvature which depends on Ω_0. We will need to use some IF() statements in Excel to compute κ via Eqn. 4. These work by assigning the cell to either the first or second value based on whether the logical test is true or false: **cell value=IF(**logical_test, value_if_true, value_if_false**).** In our case, we're going to determine whether κ is +1, 0, or -1 based on the value of Ω_0.

κ = IF(Ω_0 =1, 0, IF(Ω_0 >1, 1, -1))

Look at this logic closely because one IF statement can only decide between two choices. We need to nest two of them to decide between 3 choices. In the

spreadsheet below, N2 is set using the Excel equation:
`=IF(N1=1,0,IF(N1>1,1,-1))`

H) Curvature also has a characteristic radius of curvature, R_0. Go ahead and compute it in cell N3. Don't worry about the fact that you need to divide by zero when κ=0. For a flat universe, the curvature is infinite. This is probably the first time that #DIV/0! is the right answer.

I) You'll need more IF statements to compute $S_k(D_c)$ in column J. Take a look at Eqn. 3. The logic is:

S_k = IF($\kappa=0$, D_c, IF($\kappa=+1$, $R_0\sin(D_c/R_0)$, $R_0\sinh(D_c/R_0)$))

Debug the curvature terms, check that the $S_k = D_c$ when Ω_0=1. Check positive curvature: set Ω_0= 1.05 and check that S_k = 8275.94 Mpc in the first bin, where log(a) = -6. Then set Ω_0=0.95 and check that S_k=8845.48 Mpc in the first bin, where log(a) = -6.

When columns A-J are computed you're done with part 1! Congratulations.

Report: Recreate Figure 1 in your report. Write a few paragraphs explaining the implications of the Einstein-deSitter universe. How old is this universe? How far away is the edge of visibility? This is called the horizon distance and is defined by D_C at the time of the Big Bang. Light emitted beyond this distance has not reached planet Earth.

Include a plot of the conformal distance as a function of t_e and another as a function of the redshift from $0<z<4$. What is the conformal distance computed here? Why does it correlate with time and redshift?

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Matter Only			Ho (km/s/Mpc) = 70		c (km/s) =	2.99792E+05	D_H (Mpc) =	4.28E+03	Ω_0=	1			
2						seconds/year=	3.15581E+07	t_H (sec) =	4.41E+17	κ =	0.00E+00			
3						Mpc/km=	3.24078E-20			R_0=	#DIV/0!			
4					time since									
5			$f(a) = 2/3(a^{3/2} - 1)$	emission time	Big Bang									
6	log(a)	a	Ho(te-to)	te (sec)	age (years)	z(t)	trap Area (Mpc)	D_c (Mpc)	D_c/D_H	S_k (Mpc)	S_k/D_H	D_A/D_H	D_L/D_H	DM
7	-6	0.000001	-0.666666666	-2.93874E+17	0.000E+00	999999	0.0992	8557.8654	1.9982	8557.8654	1.9982	1.99822E-06	1998220.7	74.662
8	-5.99	1.023E-06	-0.666666666	-2.93874E+17	3.273E-01	977236.221	0.1003	8557.7662	1.9982	8557.7662	1.9982	2.04474E-06	1952713.009	74.612
9	-5.98	1.047E-06	-0.666666666	-2.93874E+17	6.660E-01	954991.586	0.1015	8557.6659	1.9982	8557.6659	1.9982	2.09235E-06	1908241.459	74.562
10	-5.97	1.072E-06	-0.666666666	-2.93874E+17	1.017E+00	933253.3008	0.1027	8557.5644	1.9982	8557.5644	1.9982	2.14106E-06	1864782.462	74.512
11	-5.96	1.096E-06	-0.666666666	-2.93874E+17	1.380E+00	912009.8394	0.1039	8557.4617	1.9981	8557.4617	1.9981	2.1909E-06	1822312.965	74.462
12	-5.95	1.122E-06	-0.666666666	-2.93874E+17	1.755E+00	891249.9381	0.1051	8557.3578	1.9981	8557.3578	1.9981	2.24191E-06	1780810.442	74.412
597	-0.1	0.7943282	-0.19470281	-8.58271E+16	6.593E+09	0.258925412	88.4073	931.5914	0.2175	931.5914	0.2175	0.172783984	0.273844161	40.346
598	-0.09	0.8128305	-0.178116978	-7.85159E+16	6.824E+09	0.230268771	89.4310	843.1841	0.1969	843.1841	0.1969	0.160029641	0.242214663	40.080
599	-0.08	0.8317638	-0.160948283	-7.09478E+16	7.064E+09	0.202264435	=(D600-D599)*0.5*($G$1/B600+$G$1/B599)*$G$3				0.1760	0.14638856	0.211595847	39.786
600	-0.07	0.851138	-0.143176244	-6.31137E+16	7.312E+09	0.174897555	91.5141	663.2866	0.1549	663.2866	0.1549	0.131819366	0.181961379	39.459
601	-0.06	0.8709636	-0.124779656	-5.50042E+16	7.569E+09	0.148153621	92.5738	571.7725	0.1335	571.7725	0.1335	0.11627899	0.153285563	39.086
602	-0.05	0.8912509	-0.105736572	-4.66098E+16	7.835E+09	0.122018454	93.6457	479.1987	0.1119	479.1987	0.1119	0.099722607	0.125543324	38.653
603	-0.04	0.9120108	-0.086024273	-3.79204E+16	8.111E+09	0.096478196	94.7301	385.5529	0.0900	385.5529	0.0900	0.082103567	0.098710199	38.130
604	-0.03	0.9332543	-0.065619242	-2.89257E+16	8.396E+09	0.071519305	95.8270	290.8228	0.0679	290.8228	0.0679	0.063373326	0.072762314	37.468
605	-0.02	0.9549926	-0.044497133	-1.96148E+16	8.691E+09	0.047128548	96.9367	194.9958	0.0455	194.9958	0.0455	0.043481373	0.047676377	36.550
606	-0.01	0.9772372	-0.022632747	-9.97676E+15	8.996E+09	0.023292992	98.0591	98.0591	0.0229	98.0591	0.0229	0.022375155	0.023429664	35.007
607	0	1	0	0.00000E+00	9.312E+09	0	0.0000	0	0.0000	0.0000	0.0000	0	0	
608	0.01	1.023293	0.023428111	1.03274E+16	9.640E+09	-0.022762779								
609	0.02	1.0471285	0.047679537	2.10177E+16	9.978E+09	-0.045007414								
610	0.03	1.0715193	0.07278321	3.20836E+16	1.033E+10	-0.066745699								
611	0.04	1.0964782	0.098769081	4.35385E+16	1.069E+10	-0.087989161								
612	0.05	1.1220185	0.125668152	5.53959E+16	1.107E+10	-0.108749062								
613	0.06	1.1481536	0.153512514	6.76700E+16	1.146E+10	-0.12903641								
614	0.07	1.1748976	0.182335387	8.03754E+16	1.186E+10	-0.148861962								
615	0.08	1.2022644	0.212171159	9.35274E+16	1.228E+10	-0.168236229								
616	0.09	1.2302688	0.243055424	1.07142E+17	1.271E+10	-0.187169484								
617	0.1	1.2589254	0.27502503	1.21234E+17	1.315E+10	-0.205671765								

Figure 3: Derived observable quantities in the Einstein-DeSitter universe.

Part 2 – General Solution to the Friedman Equation

Barbara Ryden derives the Friedman Equation, the fluid equation, and the equation of state in chapter 4. Together, they are solved in chapters 5 & 6. We will concentrate here on the general solution: (Ryden Eqn. 6.8),

$$\int_{1}^{a} \frac{da'}{\sqrt{\frac{\Omega_{r,0}}{a'^2} + \frac{\Omega_{m,0}}{a'} + \Omega_{\Lambda,0} a'^2 + (1 - \Omega_0)}} = H_0 \int_{t_0}^{t_e} dt' \qquad (13)$$

where the history of the universe is embodied in the time, t_e, and the expansion scale factor, a that is governed by the Ω parameters measured at their current epoch. This equation includes the radiation, matter, and dark energy density as well as the resulting curvature term (1 - Ω_o). The right-hand side is easily integrated giving the familiar $H_0 \int_{t_0}^{t} dt' = H_0(t_e - t_0)$. The left-hand side is more complicated and has no nice analytical solution. Furthermore, it can't be inverted into the form *a(t) = f(t)* like the Einstein-DeSitter Universe. Be sure that you understand the derivation of Eqn. 13 and what it means. Once you trust the physics behind the equation, you can use it to compute interesting facts about the universe.

General Instructions: Start with the same log(a) array as you did previously. Compute *a,* and *z* as before. Next, numerically integrate the left-hand-side of Eqn. 13 and set it equal to $H_0(t_e\text{-}t_o)$. Just like before, you'll have to be careful with the integration limits. Set a=1 when t_e=t_o and then integrate backward to t_e in the past. Check that you get the results shown in Figure 4, column F for the Ryden's Benchmark cosmology. Recreate Figure 6 below.

When the general solution for *a(t)* is done, use it to compute the distance factors from Part 1. This shouldn't require you to re-code the distance factors. To avoid problems you want to use the code from before because it's already debugged. Just replace the old $H_0(t_e\text{-}t_0)$ array with the new $H_0(t_e\text{-}t_0)$ array.

Excel Instructions:

A) Open the same Excel file that you've been using and insert a new sheet by clicking on a new tab at the bottom of the page. Copy log(a) into it from the previous sheet. Then compute a and z from log(a). Setup the parameters needed for the benchmark cosmology. Now we need to compute the integral in Eqn 13. The integrand is:

$$f(a') = \frac{1}{\sqrt{\frac{\Omega_{r,0}}{a'^2} + \frac{\Omega_{m,0}}{a'} + \Omega_{\Lambda,0} a'^2 + (1 - \Omega_0)}} \qquad (14)$$

and we compute the area under this function as we did before by finding the area in a series of trapezoids. Check your calculation using the example spreadsheet shown in Figure 4. Compute Column F by summing all the trapezoidal areas. Be careful with the integration limits; remember to add the areas into the past to find $H_0(t_e\text{-}t_0)$. Set $H_0(t_e\text{-}t_0)$=0 at t_e=t_0 then add the area of the trapezoids going upward in the table.

	A	B	C	D	E	F
1	**General Solution to Friedman Equation**				**Parameters**	
2					**Omega_M**	0.3
3					**Omega_r**	8.40E-05
4					**Omega_Lambd**	0.7
5					**Omega_tot**	1.000084
6					**Ho**	70
7						
8						
9						
10	**log(a)**	**a**	**z**	**f(a)**	**trap. area**	**Ho(te-to)**
11	-6	0.000001	999999	1.09E-04	2.57E-12	-9.64E-01
12	-5.99	1.0233E-06	977236.221	1.11E-04	2.69E-12	-9.64E-01
13	-5.98	1.0471E-06	954991.586	1.14E-04	2.81E-12	-9.64E-01
14	-5.97	1.0715E-06	933253.301	1.17E-04	2.95E-12	-9.64E-01
15	-5.96	1.0965E-06	912009.839	1.19E-04	3.09E-12	-9.64E-01
16	-5.95	1.122E-06	891249.938	1.22E-04	3.23E-12	-9.64E-01
603	-0.1	0.79432823	0.25892541	1.10E+00	2.04E-02	-2.17E-01
604	-0.09	0.81283052	0.23026877	1.10E+00	2.07E-02	-1.96E-01
605	-0.08	0.83176377	0.20226443	1.09E+00	=(B606-B605)*0.5*(D606+D605)	
606	-0.07	0.85113804	0.17489755	1.08E+00	2.13E-02	-1.55E-01
607	-0.06	0.87096359	0.14815362	1.07E+00	2.16E-02	-1.34E-01
608	-0.05	0.89125094	0.12201845	1.06E+00	2.19E-02	-1.12E-01
609	-0.04	0.91201084	0.0964782	1.05E+00	2.21E-02	-9.01E-02
610	-0.03	0.9332543	0.07151931	1.04E+00	2.24E-02	-6.80E-02
611	-0.02	0.95499259	0.04712855	1.02E+00	2.27E-02	-4.56E-02
612	-0.01	0.97723722	0.02329299	1.01E+00	2.29E-02	-2.29E-02
613	**0**	**1**	**0**	1.00E+00	**2.31E-02**	**0.00E+00**
614	0.01	1.02329299	-0.0227628	9.87E-01	2.34E-02	2.31E-02
615	0.02	1.04712855	-0.0450074	9.74E-01	2.36E-02	4.65E-02
616	0.03	1.07151931	-0.0667457	9.61E-01	2.38E-02	7.01E-02
617	0.04	1.0964782	-0.0879892	9.47E-01	2.40E-02	9.39E-02
618	0.05	1.12201845	-0.1087491	9.33E-01	2.42E-02	1.18E-01
619	0.06	1.14815362	-0.1290364	9.19E-01	2.44E-02	1.42E-01
620	0.07	1.17489755	-0.148862	9.05E-01	2.46E-02	1.67E-01
621	0.08	1.20226443	-0.1682362	8.90E-01	2.47E-02	1.91E-01
622	0.09	1.23026877	-0.1871695	8.76E-01	2.49E-02	2.16E-01
623	0.1	1.25892541	-0.2056718	8.61E-01	2.50E-02	2.41E-01

Figure 4: Spreadsheet showing the general solution to the Friedman Equation for the Benchmark Model.

B) Create another "Arbitrary" spreadsheet by copying the Einstein-deSitter spreadsheet into a new spreadsheet and relabeling the $H_0(t_e\text{-}t_0)$ column as shown in Figure 5. Plots don't copy correctly so don't bring them along. The numbers in the columns can be copied without changing anything! Now let's hook the new spreadsheet up to the General Solution. Set the $H_0(t_e\text{-}t_0)$ column equal to the general solution from your previous spreadsheet as shown below. You'll also set H_0 and Ω_0 to the value in the other spreadsheet.

To hook up a cell to another spreadsheet: click on the cell, enter =, click on the sheet tab at the bottom to bring the other sheet forward, click on the cell you want and hit enter. Go back to the original sheet and check that the cell points to the other sheet. An example of this is shown in Figure 5 where all red cells are hooked up to the spreadsheet containing the general solution to Friedman Equation. Whatever you do, DON'T CHANGE ANY EQUATIONS! THEY ALREADY WORK! It may take you a few tries, but when done properly, this part should take 2 or 3 minutes.

	A	B	C	D	E	F	G	H	I	J	K
1	Arbitrary			Ho (km/s/Mpc) =	70	c (km/s) =	2.99792E+05	D_H (Mpc) =	4.28E+03	Ω_0=	1.000084
2						seconds/year=	3.15581E+07	t_H (sec) =	4.41E+17	κ =	1.00E+00
3						Mpc/km=	3.24078E-20			R_0=	467285.555
4					time since						
5			General Solution	emission time	Big Bang						
6	log(a)	a	Ho(te-to)	te (sec)	age (years)	z(t)	rap Area (Mpc)	D_c (Mpc)	D_c/D_H	S_k (Mpc)	S_k/D_H
7	-6	0.000001	-0.9637249	-4.24821E+17	0.000E+00	999999	0.0109	13897.1083	3.2449	13895.0598	3.2444
8	-5.99	1.0233E-06	-0.9637249	-4.24821E+17	3.585E-02	977236.221	0.0111	13897.0975	3.2449	13895.0490	3.2444
9	-5.98	1.0471E-06	='General solution to Friedman'!F13		7.339E-02	954991.586	0.0114	13897.0864	3.2449	13895.0379	3.2444
10	-5.97	1.0715E-06	-0.9637249	-4.24821E+17	1.127E-01	933253.3008	0.0116	13897.0750	3.2449	13895.0265	3.2444
11	-5.96	1.0965E-06	-0.9637249	-4.24821E+17	1.538E-01	912009.8394	0.0119	13897.0633	3.2449	13895.0148	3.2444
12	-5.95	1.122E-06	-0.9637249	-4.24821E+17	1.969E-01	891249.9381	0.0122	13897.0514	3.2449	13895.0029	3.2444

Figure 5: Computation for an arbitrary cosmology. Notice how the red values are taken from the previous spreadsheet.

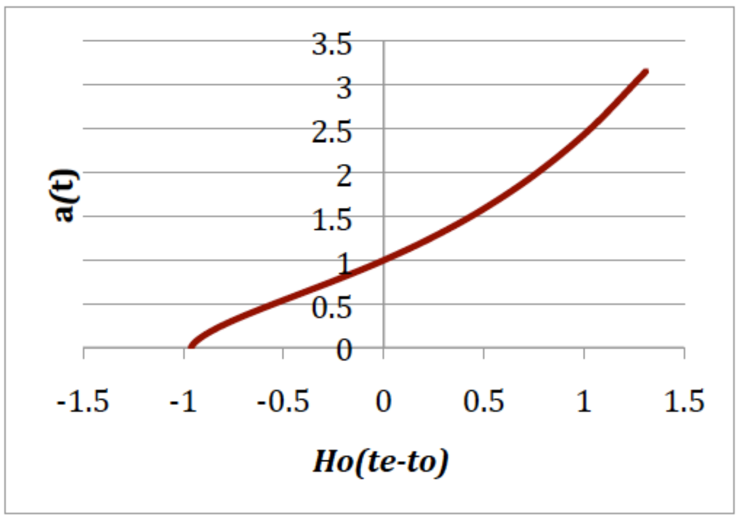


Figure 6: Scale factor as a function of time for the Benchmark Model.

Report: The following can be done without modifying the columns of your spreadsheet. From here on out you should just be changing the parameters in the spreadsheet with the General Solution and adding one plot.

A. **Benchmark Universe:** Recreate Figure 6 in your report. Verify that you compute the same age of the universe that Ryden computes in Table 6.2.

B. Now we're ready to explore several different cosmologies. Make a table with the following columns: name of cosmology, H_0, $\Omega_{r,0}$, $\Omega_{m,0}$, $\Omega_{\Lambda,0}$, Ω_0, κ, R_0, age of the universe, horizon distance. Fill it in for the Benchmark cosmology.

For each of the cosmologies below, please fill in the table, and take a screen shot of *a(t)* as shown in Figure 6 to add to your report.

C. Fill in the table for the **Flat Λ-Only** and **Flat Matter-Only** universes. The Flat Matter-Only universe is the Einstein-deSitter universe from Part 1. The numbers should come out perfectly the same! Make sure they do.

D. Another interesting universe is the **Low-Density** universe. With so much empty space in the universe, lets investigate a universe with just the atoms and radiation we're used to, i.e. $\Omega_{m,o}$ = 0.05 and $\Omega_{r,o}$ = 8.42E-5. What is the curvature of this universe? Add it to your table of universes and plot *a(t)*. In the next few weeks we will find out something surprising and exciting, that the low-density universe is not the one that we actually live in. We are going to look at observations and see that they differ from this universe.

E. The **Cold Dark Matter (CDM)** universe is one where $\Omega_{m,o}$ = 0.3 and $\Omega_{r,o}$ = 8.42E-5. This universe is inspired by the Low-density universe with the addition of dark matter. The word cold implies that the dark matter, whatever it may be, is non-relativistic. This was the leading candidate for the universe in the 1990s. Add it to your table and plot *a(t).*

F. Where were you in 1998? In 1998 astrophysicists found that the newest, more sensitive data sets wouldn't support a pure CDM universe. The data rule out a $\Omega_{\Lambda,0}$=0 universe. The **ΛCDM** universe includes a non-zero $\Omega_{\Lambda,0}$ in addition to $\Omega_{m,o}$ and $\Omega_{r,o}$. The $\Omega_{r,0}$ parameter was measured by the COBE satellite mission (1993) and extended by a theoretical calculation of the neutrino contribution. The value used in all the ΛCDM parameter sets is $\Omega_{r,0}$ = 8.42E-5.

There are 3 data sets of significant interest today.

#1) Type Ia supernovae (SNe) are used to make a modern day Hubble Plot. Please download Ref. [4] to see the modern Hubble plot in Figure 9. The results of the SNe are combined with the BAO and CMB (see Figure 11) to create the measured parameters listed in Table 10. We call the flat set of parameters for SNe+BAO+CMB by the name **ΛCDM: SNe-Union2** (H_0=69.25, $\Omega_{m,o}$=$0.279^{+0.017}_{-0.016}$, $\Omega_{\Lambda,0}$=$0.721^{+0.016}_{-0.017}$).

#2) The WMAP satellite was launched in 2001 and just released its final results last March based on 9 years of data collection. Please download Ref. [5] and look at the results in Figure 43 and the parameters listed in Table 17. We call the flat set of parameters for WMAP+eCMB+BAO+H0 the **ΛCDM: WMAP-Final** parameters. Add them to your table and plot *a(t)*.

#3) The Planck satellite was launched in 2009 and released its first results last April based on just over one year of science observations. Please download Ref. [6] and look at the results in Table 5 on page 22. We call the parameters for the Planck+WP+highL+BAO by the name **ΛCDM: Planck-2013** (H_0=67.80 $\pm$ 0.77, $\Omega_{\Lambda,0}$=0.692 $\pm$ 0.010, $\Omega_{m,o}$ = 1-$\Omega_{\Lambda,0}$ - $\Omega_{r,0}$).

How well does your age calculations agree with the published ages in the WMAP and Planck papers?

G. For the **ΛCDM: WMAP-Final** parameter set, make a full page plot of a(t) vs. age and another high resolution plot of Dc vs z. To be able to see what's going on at all length scales, you'll need to make them log plots on both the x-axis and the y-axis. You will need to use these to do the homework in upcoming chapters, so please label all the axes carefully.

H. Ryden's Benchmark model serves the important purpose of computing the expansion history pretty well in light of the rapid changes in the field. It is impossible to produce new editions of the book every time a paper is published. With this code, you can always use the most current values as they become available. Calculate the percent difference between the **Benchmark** and the **ΛCDM: WMAP-Final** table values. Then comment on whether the distance and age values in the textbook are significantly flawed or reasonably accurate.

Part 3 – Observable Astronomical Quantities

In parts 1 and 2, we computed age and distance in the universe. However, to relate it to measurements, we need to use detection equipment. There is no wooden meter stick to measure Mpc distances or stopwatch to tell us the passage of billions of years. The piece of equipment we use to learn about the universe is a telescope. It collects light. Please go back and reread the paragraphs related to Equations 7-11 to understand how the geometry of the universe affects the observed brightness and angular extent on the sky. The angular diameter distance, luminosity distance and distance modulus need to be added to your computation so that you can plot them.

General instructions: Open up your "Arbitrary" spreadsheet or program and compute the angular diameter distance and the luminosity distance normalized by the Hubble Distance. The Hubble Distance, $D_H = c/H_0$, should be pre-computed at the top of your spreadsheet. You'll also need to compute the distance modulus. Use Figure 4 to debug columns K-N. Figure 7 shows how the distance factors depend on redshift in the Einstein-DeSitter Universe.

Report: Recreate Figure 7 for the Einstein-DeSitter and Benchmark Universes. Do galaxies appear smaller at greater redshifts? Explain carefully using Eqn. 7 and the plots. Does the measured brightness go down for objects at greater redshifts? Explain using Eqn. 9 and the plots. Does this universe continue to expand forever or does it start to contract at some time in the future?

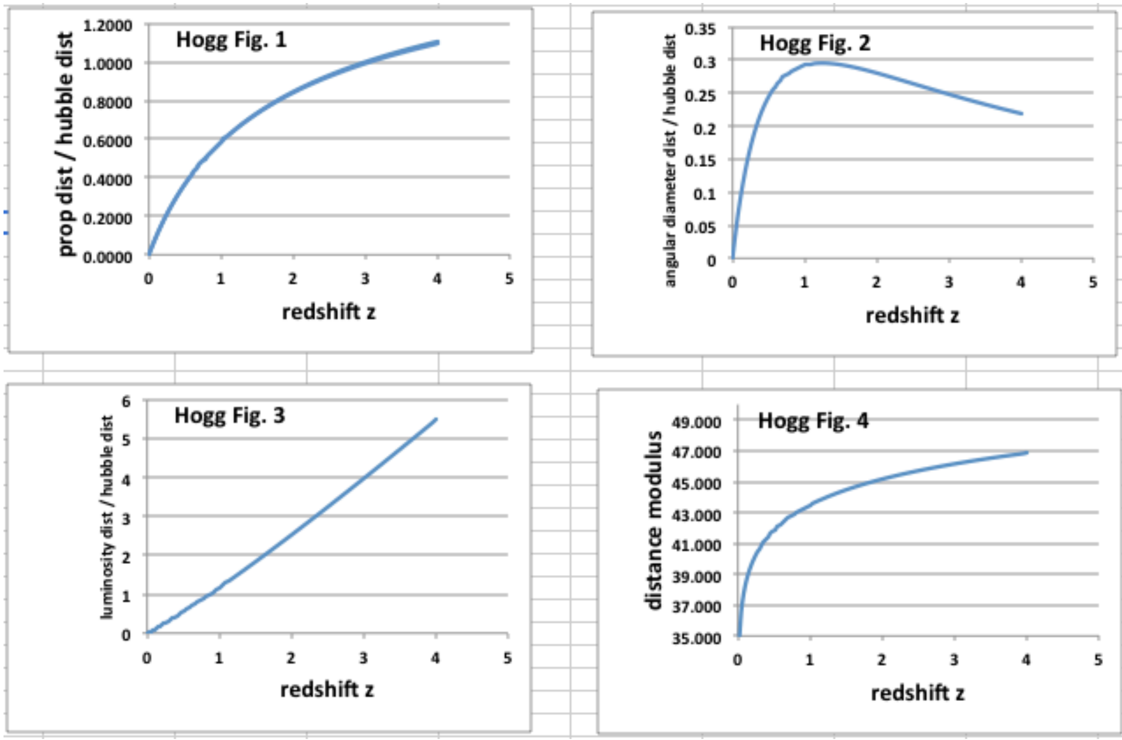


Figure 7: Plots of observational distance factors in the Einstein-DeSitter universe. These are recreations of Figures in reference [3] by David Hogg.

Part 4 – Ruling out the Low-density, CDM, and Matter-only universes

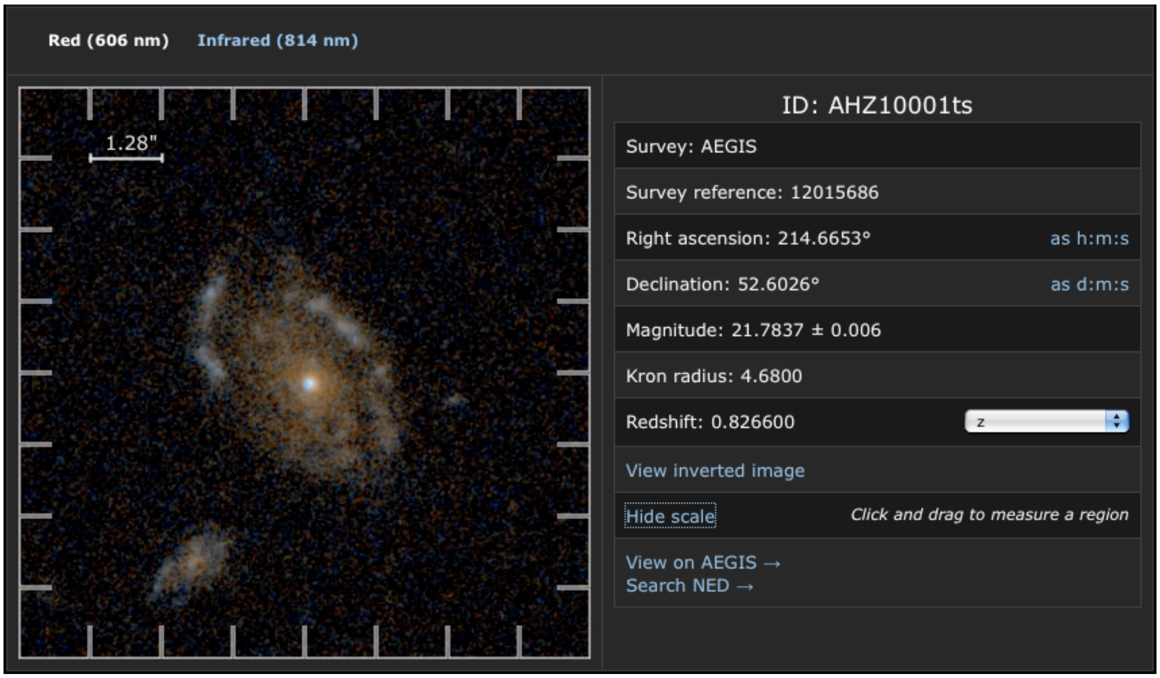
Ryden's Figure 7.5 shows how data are used to constrain the parameters of the theory that models the expansion of the universe. Many new type Ia supernovae have been discovered since 1999. The most recent presentation of these data are in Figure 9 on page 19 in reference [4].

A. Explain the differences between the Ryden's Figure 7.5 and Figure 9 in the new supernova paper [4].
B. Use all 8 cosmologies explored in Part 1 to create theory curves like the upper panel in Figure 9 in reference [4]. (*This problem shouldn't require any additional coding. You can copy and paste results from your spreadsheet into a new spreadsheet for plotting.*)
C. Does a supernova in a universe with more matter appear dimmer or brighter? Does a supernova in a universe with more dark energy appear dimmer or brighter?
D. Subtract the other theory predictions from the **ΛCDM: SNe-Union2** prediction to recreate the lower panel in Figure 9 of reference [4].
E. Theoretical predictions exist for many things that do not actually exist. The data tell us what actually exists. Which of the theories we've explored are NOT consistent with the data? How do you feel about saying that the data rule out these possibilities?

Part 5 – Measuring the size of distant galaxies

1. The image below was extracted from the Galaxy Zoo Hubble project [6].

A) How big is this galaxy in its rest frame at the time it emitted the light captured here in the Hubble telescope? (Please use the **ΛCDM: WMAP-Final** parameters from Part 2 to answer the question. You'll need to use D_C or $S_K(D_C)$ with Eqns. 7 & 8.)
B) How big is the satellite galaxy?
C) How far apart are the two galaxies?
D) How does the size of the large galaxy compare to the size of the Milky Way?
E) Does the Milky Way have any satellite galaxies? If so, how big are they?



2. Now find a galaxy of your own and measure it. Go to the Galaxy Zoo Hubble project [6]: http://www.galaxyzoo.org/. Classify 20 galaxies and as you do so, investigate the ones that look interesting. To investigate the galaxy, you'll have to say that you want to discuss it. Find the redshift and angular scale for the image in the detailed information available. Be careful with the redshift because there is also a "z" light filter and numbers >2 are highly unlikely. Choose a galaxy with a z>0.1.

A) Include a screen shot of the galaxy and it's information in your report.
B) How big is the galaxy at the time it emitted the light captured by the telescope?
C) How does it compare to the Milky Way galaxy? How does it compare to other galaxies in the Local Group of galaxies, like the Large and Small Magellanic Clouds?

References:

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2. http://www.nobelprize.org/nobel_prizes/physics/laureates/2011/
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4. R. Amanullah, et al. (Supernova Cosmology Project), *Spectra and Light Curves of Six Type Ia Supernovae at 0.511 < z < 1.12 and the Union2 Compilation*, Astrophys.J.716:712-738, (2010), http://arxiv.org/abs/1004.1711.
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7. C.J. Lintott, et al., *Galaxy Zoo: Morphologies derived from visual inspection of galaxies from the Sloan Digital Sky Survey*, Mon. Not. Roy. Astron. Soc. 389:1179, (2008), www.galazyzoo.org, www.sdss.org.