Mid-term exam, FYS - 4110, Autumn 2013

Cand. nr.: 7

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a)

We have the following expression for the density-matrix for a system composed of two two-level systems

$$\hat{\rho} = \frac{1}{4}[\mathbb{1} \otimes \mathbb{1} + \vec{a} \cdot \vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{b} \cdot \vec{\sigma} + \sum_{ij} c_{ij} \sigma_i \otimes \sigma_j], \tag{1}$$

where a_i, b_j and c_{ij} are all real.

We want to find the expression for $\hat{\rho}^2$ on the same form. We get

$$\begin{aligned}
\hat{\rho}^2 &= \frac{1}{16}[\mathbb{1} \otimes \mathbb{1} + 2(\vec{a} \cdot \vec{\sigma} \otimes \mathbb{1}) + 2(\mathbb{1} \otimes \vec{b} \cdot \vec{\sigma}) + 2\sum_{ij} c_{ij} \sigma_i \otimes \sigma_j \\
&+ (\vec{a} \cdot \vec{\sigma})^2 \otimes \mathbb{1} + \mathbb{1} \otimes (\vec{b} \cdot \vec{\sigma})^2 + 2(\vec{a} \cdot \vec{\sigma} \otimes \vec{b} \cdot \vec{\sigma}) \\
&+ \sum_{ijk} a_k c_{ij} \underbrace{(\sigma_k \sigma_i + \sigma_i \sigma_k)}_{2\delta_{ki}} \otimes \sigma_j + \sum_{ijk} b_k c_{ij} \sigma_i \otimes \underbrace{(\sigma_k \sigma_j + \sigma_j \sigma_k)}_{2\delta_{kj}} \\
&+ \sum_{ijkl} c_{ij} c_{kl} \underbrace{\sigma_i \sigma_k}_{(\delta_{ik} + i\sum_m \epsilon_{ikm} \sigma_m)} \otimes \underbrace{\sigma_j \sigma_l}_{(\delta_{jl} + i\sum_n \epsilon_{jln} \sigma_n)} \quad]
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{1}{16}\Bigg[\left(1+|\vec{a}|^2+|\vec{b}|^2+\sum_{ij} c_{ij}^2\right)\ \mathbb{1}\otimes\mathbb{1} \\
+ \quad & \left(2\sum_i a_i + 2\sum_{ij} c_{ij}b_j + i\underbrace{\sum_{ijkl} c_{jk}c_{lk}\epsilon_{jli}}_{=0}\right)\ \sigma_i\otimes\mathbb{1} \\
+ \quad & \left(2\sum_i b_i + 2\sum_{ij} c_{ij}a_j + i\underbrace{\sum_{ijkl} c_{jk}c_{jl}\epsilon_{kli}}_{=0}\right)\ \mathbb{1}\otimes\sigma_i \\
+ \quad & \sum_{ij}\left(2(a_ib_i+c_{ij}) - \sum_{klmn} c_{kl}c_{mn}\epsilon_{kmi}\epsilon_{lnj}\right)\ \sigma_i\otimes\sigma_j\Bigg]
\end{aligned}
$$

The two expressions are equal to zero since the terms cancel eachother one by one, if we for example switch j and l in the first of these terms, we get the same expression, just with a minus sign due to the anti-symmetry of the Levi-Civita tensors.

We finally get

$$
\begin{aligned}
= \quad & \frac{1}{16}\Bigg[\left(1+|\vec{a}|^2+|\vec{b}|^2+\sum_{ij} c_{ij}^2\right)\ \mathbb{1}\otimes\mathbb{1} \\
+ \quad & \left(2\sum_i a_i + 2\sum_{ij} c_{ij}b_j\right)\ \sigma_i\otimes\mathbb{1} \\
+ \quad & \left(2\sum_i b_i + 2\sum_{ij} c_{ij}a_j\right)\ \mathbb{1}\otimes\sigma_i \\
+ \quad & \sum_{ij}\left(2(a_ib_i+c_{ij}) - \sum_{klmn} c_{kl}c_{mn}\epsilon_{kmi}\epsilon_{lnj}\right)\ \sigma_i\otimes\sigma_j\Bigg]. \qquad (2)
\end{aligned}
$$

We also want to find the reduced density matrices $\hat{\rho}_A$ and $\hat{\rho}_B$

$$
\begin{aligned}
\hat{\rho}_A = Tr_B(\hat{\rho}) &= \frac{1}{4}\left(\mathbb{1}\ Tr(\mathbb{1}) + \vec{a}\cdot\vec{\sigma}\ Tr(\mathbb{1}) + \sim \underbrace{Tr(\sigma_i)}_{=0}\right) \\
&= \frac{1}{2}\left(\mathbb{1}+\vec{a}\cdot\vec{\sigma}\right).
\end{aligned}
$$

Similarly we get

$$\hat{\rho}_B = \frac{1}{2}\left(\mathbb{1} + \vec{b}\cdot\vec{\sigma}\right).$$

We want to find the square of the reduced density-matrices, we get

$$\hat{\rho}_A^2 = \frac{1}{4}\left(\mathbb{1} + 2\ \vec{a}\cdot\vec{\sigma} + (\vec{a}\cdot\vec{\sigma})^2\right) = \frac{1}{4}\left((1+|\vec{a}|^2)\ \mathbb{1} + 2\ \vec{a}\cdot\vec{\sigma}\right).$$

We see now that $|\vec{a}| = 1$ is the special case of a pure state where $\rho_A = \rho_A^2$. We get the same expression for system B.

b)

If we expand the density matrix in the basis of its eigenstates we get a diagonal matrix

$$\hat{\rho} = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

Since p_k is a probability we have that $p_k \leq 1$ and $\sum_k p_k = 1$. This way we get

$$Tr(\hat{\rho}^2) = \sum_k p_k^2 \leq \sum_k p_k = 1.$$

For a pure state we have that one of the p_k's are equal to 1 and the others are eqal to zero, in this case we have

$$Tr(\hat{\rho}^2) = 1^2 = 1 = Tr(\hat{\rho}).$$

When we take trace of the squared density matrix we see that the only part of eqn 2 that contributes is the part proportional to $\mathbb{1}\otimes\mathbb{1}$. Writing this out we get

$$Tr(\hat{\rho}^2) = \frac{1}{16}\left(1 + |\vec{a}|^2 + |\vec{b}|^2 + \sum_{ij} c_{ij}^2\right) Tr(\mathbb{1}\otimes\mathbb{1}) = \frac{1}{4}\left(1 + |\vec{a}|^2 + |\vec{b}|^2 + \sum_{ij} c_{ij}^2\right).$$

Setting this less than or equal to one we get

$$\left(1 + |\vec{a}|^2 + |\vec{b}|^2 + \sum_{ij} c_{ij}^2\right) \leq 4 \Rightarrow$$

$$\left(|\vec{a}|^2 + |\vec{b}|^2 + \sum_{ij} c_{ij}^2\right) \leq 3.$$

c)

We want to find the requirements on a, b and c in order for the system to be in a tensor-product state. A tensor-product density matrix can be written on the following form

$$\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B = \frac{1}{2}\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right) \otimes \frac{1}{2}\left(\mathbb{1} + \vec{b}\cdot\vec{\sigma}\right)$$
$$= \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \vec{a}\cdot\vec{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\vec{b}\cdot\vec{\sigma} + \vec{a}\cdot\vec{\sigma}\otimes\vec{b}\cdot\vec{\sigma}\right).$$

Comparing with eqn 1 we see that this implies

$$c_{ij} = a_i b_j.$$

We also want to verify the requirements on a, b and c in the case of a maximally entangled pure state. For maximum entanglement we need

$$\hat{\rho}_A = \hat{\rho}_B = \frac{1}{2}\mathbb{1},$$

this imlies that

$$\vec{a} = \vec{b} = \vec{0}.$$

In addition, we have a pure state, so we have

$$\hat{\rho} = \hat{\rho}^2.$$

Since a and b are zero this means that (from eqn 1 and 2)

$$\frac{1}{16}\left(1 + \sum_{ij} c_{ij}^2\right) = \frac{1}{4}.$$

and

$$\frac{1}{16}\left(2c_{ij} - \sum_{klmn} c_{kl}c_{mn}\epsilon_{kmi}\epsilon_{lnj}\right) = \frac{1}{4}c_{ij}.$$

The first equation gives us

$$\sum_{ij} c_{ij}^2 = 3.$$

The second gives us

$$-\sum_{klmn} c_{kl}c_{mn}\epsilon_{kmi}\epsilon_{lnj} = 2c_{ij}.$$

d)

For the composite system we use the following representation:

$$|++\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, |+-\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, |-+\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, |--\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

We then represent the two Bell-states in the follwing way

$$|B1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \quad |B2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},$$

we get

$$\hat{\rho}_{B1} = \frac{1}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\1 & 0 & 0 & 1\end{pmatrix}.$$

We also have these expressions

$$\sigma_z \otimes \sigma_z = \begin{pmatrix}1 & 0 & 0 & 0\\0 & -1 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & 1\end{pmatrix},$$

$$\sigma_x \otimes \sigma_x = \begin{pmatrix}0 & 0 & 0 & 1\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\1 & 0 & 0 & 0\end{pmatrix},$$

$$\sigma_y \otimes \sigma_y = \begin{pmatrix}0 & 0 & 0 & -1\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\-1 & 0 & 0 & 0\end{pmatrix}.$$

We can now find an expression for $\hat{\rho}_{B1}$ in terms of these matrices

$$\hat{\rho}_{B1} = \frac{1}{4}\left(\mathbb{1} \otimes \mathbb{1} + \sigma_z \otimes \sigma_z + \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y\right).$$

For $\hat{\rho}_{B2}$ we get

$$\hat{\rho}_{B2} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

This becomes

$$\hat{\rho}_{B2} = \frac{1}{4}\left(\mathbb{1} \otimes \mathbb{1} + \sigma_z \otimes \sigma_z - \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y\right).$$

We see that in both our two Bell-states the coefficients c_{ij} are only diagonal. Lets see if the requirements we found in c), for maximally entangled pure states, are fulfilled by our two Bell-states. We will look at ρ_{B1} first.

We see that $c_{11} = -1$ and $c_{22} = c_{33} = 1$. We easily see that the first requirement is fulfilled

$$\sum_{ij} c_{ij}^2 = -1^2 + 1^2 + 1^2 = 3.$$

This holds equally for ρ_{B2} as well.

The other requirement is a bit more tricky

$$-\sum_{klmn} c_{kl} c_{mn} \epsilon_{kmi} \epsilon_{lnj} = 2c_{ij}.$$

If we use the fact that c is diagonal, we see that only the terms in the left hand side sum where $k = l$ and $m = n$ will be nonzero.

$$-\sum_{km} c_{kk} c_{mm} \epsilon_{kmi} \epsilon_{kmj} = 2c_{ij}.$$

We can now replace the two Levi-civita tensors by δ_{ij} and the requirement that $k \neq m$. Since the left hand side now is symmetric in k and m we can restrict ourself further to the case $k < m$ if we remove a factor of two on the right hand side.

$$-\sum_{km} c_{kk} c_{mm} \delta_{ij} = c_{ij}.$$

It is now clear that the left hand side is also diagonal in i, j like it should be. Now, with the requirement $k < m$, we only get 3 equations

$$c_{11} c_{22} = -c_{33}, \tag{3}$$

$$c_{11}c_{33} = -c_{22}, \tag{4}$$

$$c_{22}c_{33} = -c_{11}. \tag{5}$$

It is now easy to see that both our Bell-states satisfy these equations.

e)

We are given the time dependent state

$$|\psi_1(t)\rangle = \cos\omega t|B1\rangle + \sin\omega t|B2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} c+s \\ 0 \\ 0 \\ c-s \end{pmatrix},$$

where we introduce the notation $c = \cos\omega t$, and $s = \sin\omega t$.

We want to find the corresponding density matrix $\hat{\rho}_1(t)$

$$\hat{\rho}_1(t) = \frac{1}{2}\begin{pmatrix} c+s \\ 0 \\ 0 \\ c-s \end{pmatrix}\begin{pmatrix} c+s & 0 & 0 & c-s \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (c+s)^2 & 0 & 0 & c^2-s^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c^2-s^2 & 0 & 0 & (c+s)^2 \end{pmatrix}.$$

Multiplying out we get

$$\hat{\rho}_1(t) = \frac{1}{2}\begin{pmatrix} 1+2cs & 0 & 0 & c^2-s^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c^2-s^2 & 0 & 0 & 1-2cs \end{pmatrix}.$$

We now observe that the following combination is what we need (in addition to ones used earlier)

$$\frac{1}{2}(\mathbb{1}\otimes\sigma_z + \sigma_z\otimes\mathbb{1}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We now get

$$\hat{\rho}_1(t) = \frac{1}{4}\left[\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\sigma_z + 2cs(\mathbb{1}\otimes\sigma_z + \sigma_z\otimes\mathbb{1}) + (c^2 - s^2)(\sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y\right].$$

Since this is a pure quantum state, the entropy of one of the subsystems is taken as a measure of the entanglement between the two subsystems. We therfore need to find $\hat{\rho}_{1A}(t)$

$$\hat{\rho}_{1A} = Tr_B(\hat{\rho}_1) = \frac{1}{2}(\mathbb{1} + 2cs\sigma_z).$$

We see that the eigenvalues are given as $\frac{1}{2} \pm cs$. This gives the following entropy

$$S_1 = -\left[\left(\frac{1}{2}+cs\right)\log\left(\frac{1}{2}+cs\right) + \left(\frac{1}{2}-cs\right)\log\left(\frac{1}{2}-cs\right)\right].$$

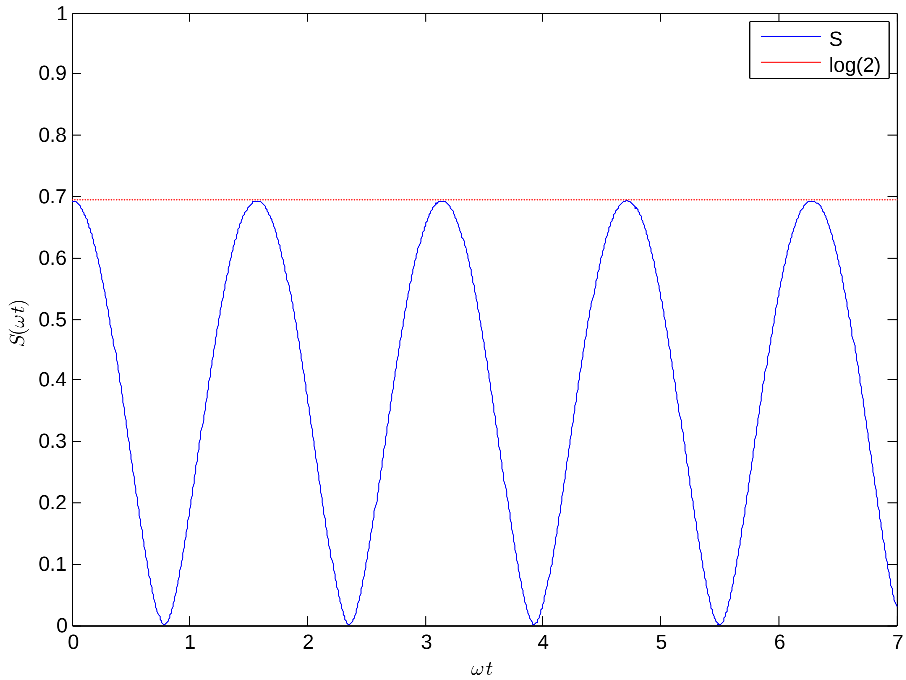

Figure 1: Plot of the entropy of the subsystem 1A. This is used as a measure of the entanglement between the two subsystems 1A and 1B. (The system 1B is not to be confused with the Bell-state $|B1\rangle$.)

f)

We have another density-matrix given as a sum of the density-matrices for the Bell-states $|B1\rangle$ and $|B2\rangle$

$$\hat{\rho}_2(t) = c^2|B1\rangle\langle B1| + s^2|B2\rangle\langle B2|.$$

We see that the eigenvalues of the density matrix is s^2 and c^2. This gives us the entropy of the whole system

$$S_2 = -c^2 \log c^2 - s^2 \log s^2.$$

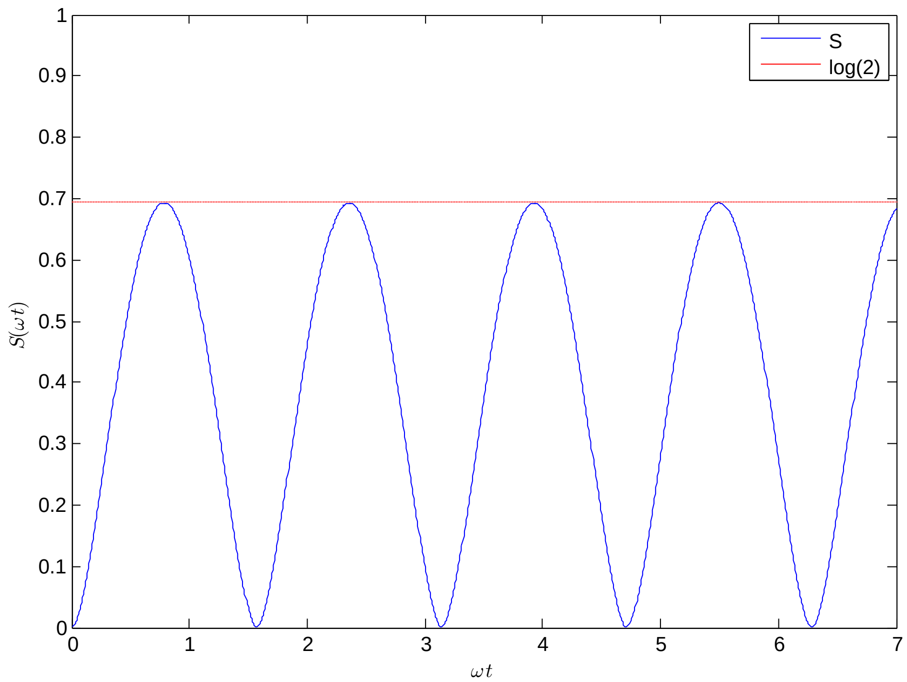

Figure 2: Plot of the entropy of the total system of $\hat{\rho}_2$.

We also want to find the entropy of the subsystems $\hat{\rho}_{2A}$ and $\hat{\rho}_{2B}$

$$\hat{\rho}_{2A} = Tr_B(\hat{\rho}_2) = c^2 Tr_B(\hat{\rho}_{B1}) + s^2 Tr_B(\hat{\rho}_{B2}) = (c^2 + s^2)\frac{1}{2}\mathbb{1} = \hat{\rho}_{2B}.$$

We see that the subsystems have maximal entropy at all times, but this is not nececcarily a good measure of the entanglement of the two subsystems, since $\hat{\rho}_2(t)$ is not a pure state. In fact, as we will se in part g), the two systems can be completely unentangled.

g)

We will now look at the two density matrices at the time $\omega t = \frac{\pi}{4}$ and show that they are seperable (writable as a sum of tensor-product states). At this time $s = c = \frac{1}{\sqrt{2}}$. We get

$$\hat{\rho}_1(\frac{\pi}{4\omega}) = \frac{1}{4}\left[\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\sigma_z + \mathbb{1}\otimes\sigma_z + \sigma_z\otimes\mathbb{1}\right].$$

We see that this is simply a tensor-product state

$$\hat{\rho}_1(\frac{\pi}{4\omega}) = \frac{1}{2}\left[\mathbb{1} + \sigma_z\right]\otimes\frac{1}{2}\left[\mathbb{1} + \sigma_z\right].$$

For $\hat{\rho}_2$ we get

$$\begin{aligned}
\hat{\rho}_2(\frac{\pi}{4\omega}) &= \frac{1}{2}(|B1\rangle\langle B1| + |B2\rangle\langle B2| \\
&= \frac{1}{4}\left[(|++\rangle + |--\rangle)(\langle ++| + \langle --|) + (|++\rangle - |--\rangle)(\langle ++| - \langle --|)\right] \\
&= \frac{1}{2}\left[|++\rangle\langle ++| + |--\rangle\langle --|\right] \\
&= \frac{1}{2}\left[(|+\rangle\langle +|)\otimes(|+\rangle\langle +|) + (|-\rangle\langle -|)\otimes(|-\rangle\langle -|)\right].
\end{aligned}$$

We see that $\hat{\rho}_2$ is just a sum of two tensor-product states.

This means that both $\hat{\rho}_1(\frac{\pi}{4\omega})$ and $\hat{\rho}_1(\frac{\pi}{4\omega})$ are separable. This means that they contain only classical correlations (no entanglement).

2

We are now looking at a system where an atom is kept in a reflecting cavity and couples to the electromagnetic field. The coupling of the atom to one of the modes of the EM field is strong, and we can neglect the coupling to the other modes. This is so because the energy-difference between the groundstate and the first exited state in the atom corresponds to the same frequency as one of the modes of the EM field. Thus, the atom can emit or absorb a photon with the exact frequency of the EM-mode.

The Hamiltonian-operator for this system is given by

$$\hat{H} = \frac{1}{2}\hbar\omega\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\lambda(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) - i\gamma\hbar\hat{a}^\dagger\hat{a},$$

where the Pauli matrices operate on the two-level atom system (groundstate and first exited state), and the creation and annihilation operators operate on the electromagnetic fields (creates and destroys photons). ω is the frequency of the EM-mode that is coupled to the atom. λ and γ are real parameters with units of frequency. The relation between the different parameters is assumed to be $\omega > \lambda > \gamma$.

We consider three different states of the total system, $|g,0\rangle, |g,1\rangle$ and $|e,0\rangle$. g and e represents the ground state and the exited state of the atom, while 0 and 1 is the number of photons in the relevant node of the electromagnetic field.

a)

We look now at the two dimensional subspace spanned by the two states, $|g,1\rangle$ and $|e,0\rangle$. These are the states where there is one "unit" of energy in the system. The Hamiltonian is then given by

$$\hat{H} = \begin{pmatrix} \langle e,0|\hat{H}|e,0\rangle & \langle e,0|\hat{H}|g,1\rangle \\ \langle g,1|\hat{H}|e,0\rangle & \langle g,1|\hat{H}|g,1\rangle \end{pmatrix}.$$

To find these matrix elements we can look at how $\hat{H}$ acts on the two states

$$\hat{H}|e,0\rangle = \frac{1}{2}\hbar\omega|e,0\rangle + \frac{1}{2}\hbar\lambda|g,1\rangle,$$

$$\hat{H}|g,1\rangle = \left(\frac{1}{2}\hbar\omega - i\gamma\hbar\right)|g,1\rangle + \frac{1}{2}\hbar\lambda|e,0\rangle.$$

This gives us the following matrix

$$\hat{H} = \frac{1}{2}\hbar\begin{pmatrix} \omega & \lambda \\ \lambda & \omega - 2i\gamma \end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix} \omega - i\gamma & 0 \\ 0 & \omega - i\gamma \end{pmatrix} + \frac{1}{2}\hbar\begin{pmatrix} i\gamma & \lambda \\ \lambda & -i\gamma \end{pmatrix}.$$

b)

We now want to find the time-evolution operator, defined in the usual way as

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}.$$

In order to find an expression for $\hat{U}$, it is useful to obeserve that we can write the Hamiltonian matrix on a different form

$$\hat{H} = \frac{1}{2}\hbar\mathbb{1} + \hbar\left(\frac{1}{2}i\gamma\sigma_z + \frac{1}{2}\lambda\sigma_x\right) = \frac{1}{2}\hbar\mathbb{1} + \hbar\left(\vec{\Omega}\cdot\vec{\sigma}\right),$$

where $\vec{\Omega} = \frac{1}{2}(\lambda, 0, i\gamma)$. We also define $\Omega = \frac{1}{2}\sqrt{\lambda^2 - \gamma^2}$.

Lets look at how we can rewrite the exponential

$$e^{-i\hat{A}} = \exp\left(\sum_{n=0}^{\infty}(-i)^n \frac{\hat{A}^n}{n!}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{\hat{A}^{2n}}{2n!} - i\sum_{n=0}^{\infty}(-1)^n \frac{\hat{A}^{2n+1}}{(2n+1)!}.$$

Setting $\hat{A} = t\vec{\Omega} \cdot \vec{\sigma}$ we get

$$\hat{A}^{2n} = (\hat{A}^2)^n = (\frac{t^2}{4}(\lambda^2 - \gamma^2)\mathbb{1})^n = \mathbb{1}(\Omega t)^{2n},$$

using this we get

$$\hat{A}^{2n+1} = \hat{A}(\hat{A}^2)^n = \hat{A}(\Omega t)^{2n} = \frac{\hat{A}}{\Omega t}(\Omega t)^{2n+1}.$$

Inserting back in the Taylor-expansion we get

$$e^{-i\hat{A}} = \sum_{n=0}^{\infty}(-1)^n \frac{(\Omega t)^{2n}}{2n!}\mathbb{1} - i\frac{\hat{A}}{\Omega t}\sum_{n=0}^{\infty}(-1)^n \frac{(\Omega t)^{2n+1}}{(2n+1)!} = \cos(\Omega t)\mathbb{1} - i\frac{\hat{A}}{\Omega t}\sin(\Omega t).$$

We see that this gives us

$$\hat{U}(t) = e^{-\frac{i}{2}(\omega - i\gamma)t}\left(\cos(\Omega t)\mathbb{1} - i(\frac{\vec{\Omega}}{\Omega} \cdot \vec{\sigma})\sin(\Omega t)\right).$$

c)

We now want to find the time evolution of the system if we start in the state $|\psi(0)\rangle = |e, 0\rangle$. First we write the time evolution operator explicitly on matrix form

$$\hat{U}(t) = e^{-\frac{i}{2}(\omega - i\gamma)t}\begin{pmatrix} \cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t & -\frac{i\lambda}{2\Omega}\sin\Omega t \\ -\frac{i\lambda}{2\Omega}\sin\Omega t & \cos\Omega t - \frac{\gamma}{2\Omega}\sin\Omega t \end{pmatrix}.$$

We can now find the time dependent state $|\psi(t)\rangle$

$$\begin{aligned} |\psi(t)\rangle &= e^{-\frac{i}{2}(\omega - i\gamma)t}\begin{pmatrix} \cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t & -\frac{i\lambda}{2\Omega}\sin\Omega t \\ -\frac{i\lambda}{2\Omega}\sin\Omega t & \cos\Omega t - \frac{\gamma}{2\Omega}\sin\Omega t \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= e^{-\frac{i}{2}(\omega - i\gamma)t}\begin{pmatrix} \cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t \\ -\frac{i\lambda}{2\Omega}\sin\Omega t \end{pmatrix}. \end{aligned}$$

d)

We will now look at the density matrix of the state $|\psi(t)\rangle$ and see how to compensate for the fact that the norm of the state is not conserved. We define

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|,$$

and

$$\hat{\rho}_{cav}(t) = \hat{\rho}(t) + f(t)|g,0\rangle\langle g,0|.$$

Since the norm of $|\psi(t)\rangle$ is not conserved, we will have to find a time-dependent normalization factor in order for $\hat{\rho}_{cav}(t)$ to have unit norm. We get

$$\begin{aligned}
\langle\psi(t)|\psi(t)\rangle &= e^{-\gamma t}\left(\cos\Omega t + \tfrac{\gamma}{2\Omega}\sin\Omega t \quad \tfrac{i\lambda}{2\Omega}\sin\Omega t\right)\begin{pmatrix}\cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t \\ -\frac{i\lambda}{2\Omega}\sin\Omega t\end{pmatrix} \\
&= e^{-\gamma t}\left(\cos^2(\Omega t) + \frac{\gamma^2+\lambda^2}{4\Omega^2}\sin^2(\Omega t) + \frac{\gamma}{\Omega}\sin\Omega t\cos\Omega t\right) \equiv N(t).
\end{aligned}$$

The normalized form of $|\psi(t)\rangle$ is then

$$|\psi_N(t)\rangle = \frac{|\psi(t)\rangle}{\sqrt{N(t)}}.$$

We also get

$$\hat{\rho}(t) = N(t)|\psi_N(t)\rangle\langle\psi_N(t)|,$$

We can now rewrite our expression for $\hat{\rho}_{cav}(t)$

$$\hat{\rho}_{cav}(t) = N(t)|\psi_N(t)\rangle\langle\psi_N(t)| + f(t)|g,0\rangle\langle g,0|.$$

We now see that $\hat{\rho}_{cav}(t)$ has two eigenvalues, $N(t)$ and $f(t)$, but for the norm to be conserved we need that

$$\begin{aligned}
N(t) + f(t) &= 1 \Rightarrow \\
f(t) &= 1 - N(t) \\
&= 1 - e^{-\gamma t}\left(\cos^2(\Omega t) + \frac{\gamma^2+\lambda^2}{4\Omega^2}\sin^2(\Omega t) + \frac{\gamma}{\Omega}\sin\Omega t\cos\Omega t\right).
\end{aligned}$$

I think it is very reasonable to include the addition to the density operator, since it represents something important about our system. When there is a unit of energy in the system (the atom is exited, or there is a photon in the EM -mode), the system is in an unstable state. This instability lies in the fact that the energy can be lost to the outside. In our model the state without a unit of energy in the system, $|g, 0\rangle$, is the "true" ground state, and any state will eventually end up in this state.

If γ is very small the system can oscillate between the two states with one unit of energy for a long time, and if we are only interested in small timescales it might be reasonable to neglect the addition to the density operator. Without any such restriction, the additional term is necessary to describe the system fully, it will act much like a damping term in the classical harmonic oscillator.

e)

The probability of finding an the atom in the exited state is given by the square of the overlap between the state $|\psi_N(t)\rangle$ and $|e, 0\rangle$, times the corresponding eigenvalue of the density operator. We get

$$\begin{aligned} P(e) &= N(t)|\langle e, 0|\psi_N(t)\rangle|^2 = |\langle e, 0|\psi(t)\rangle|^2 \\ &= e^{-\gamma t}\left(\cos^2(\Omega t) + \frac{\gamma^2}{4\Omega^2}\sin^2(\Omega t) + \frac{\gamma}{\Omega}\sin\Omega t\cos\Omega t\right). \end{aligned}$$

For the probability of finding the atom in the ground state we have to think a bit more carefully. There are two ways for the atom to be in the ground state, either $|g, 1\rangle$ or $|g, 0\rangle$, however we do not add the amplitudes and then square them, because they represent two different eigenstates of the density operator, we just add the individual probabilities. (An easier way to do this would be to say that it is just the probability of not being in the exited state. $P(g) = 1 - P(e)$.)

$$P(g) = N(t)|\langle g, 1|\psi_N(t)\rangle|^2 + f(t) = |\langle g, 1|\psi(t)\rangle|^2 + f(t) = e^{-\gamma t}\left(\frac{\lambda^2}{4\Omega^2}\sin^2(\Omega t)\right) + f(t).$$

The probability of finding a photon in the cavity is easier since this possibiliy is only in one of the eigenstates of the density operator. We get

$$P(1) = N(t)|\langle g, 1|\psi_N(t)\rangle|^2 = e^{-\gamma t}\left(\frac{\lambda^2}{4\Omega^2}\sin^2(\Omega t)\right).$$

We see all these probabilities plotted in figure 3.

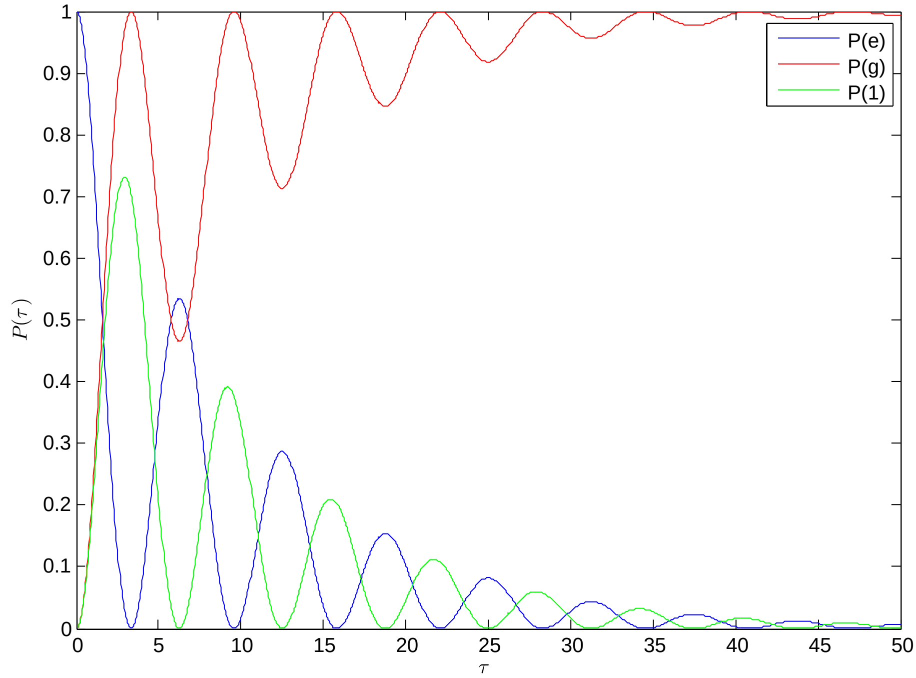

Figure 3: Plot of the different probabilities. $P(e)$ is the probability of finding the atom in the exited state. $P(g)$ is the probability of finding the atom in the ground state. $P(1)$ is the probability of finding a photon in the cavity. Parameters used: $\lambda = 1, \gamma = \frac{1}{10}, \tau = \lambda t$.

f)

As we saw in part 2 d) $\hat{\rho}_{cav}$ has two eigenvalues $f(t)$ and $N(t) = 1 - f(t)$. Now we look at the cavity-system as a sub-system of a larger system in a pure quantum state. In this case the entropy of one of the subsystems is used as a measure of the entanglement between the sub-systems. Since we already have the two eigenvalues of the cavity-system we use this to calculate the entropy. We get

$$\begin{aligned} S_{cav} &= -N(t)\log N(t) - f(t)\log f(t) \\ &= -(1 - f(t)\log(1 - f(t)) - f(t)\log f(t). \end{aligned}$$

We see this entropy plotted in figure 4.

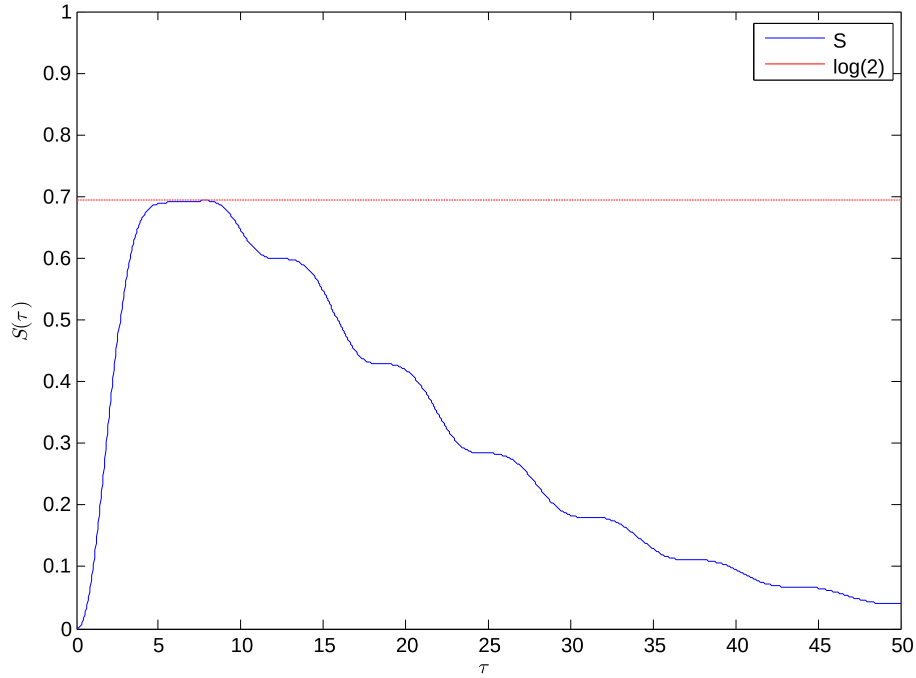

Figure 4: Plot of the entanglement entropy between the cavity-system and the external EM -field. Parameters used: $\lambda = 1, \gamma = \frac{1}{10}, \tau = \lambda t$.