Mid-term exam, FYS-4170, 2013

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1

a)

We have the Dirac equation for a charged particle in an electromagnetic field

$$[\vec{\alpha}\cdot\vec{\pi} + m\beta + V]\,\Psi = i\frac{\partial\Psi}{\partial t}. \tag{1}$$

We have $\vec{\pi} = \vec{p} - e\vec{A}(x)$, where $\vec{p} = -i\vec{\nabla}$, and $V = eA_0$.

We assume we can separate the wavefunction, Ψ, into a time dependent and a time independent part in the following way

$$\Psi(x) = e^{-iEt}\Psi_E(\vec{x}) = e^{-iEt}\begin{pmatrix}\phi(\vec{x})\\ \eta(\vec{x})\end{pmatrix}.$$

We split the energy into two parts, $E = m + E_{NR}$. We also work in the non-relativistic limit where $|E_{NR}| \ll m$ and $|V| \ll m$.

We easily find the time derivative

$$i\frac{\partial\Psi}{\partial t} = E\Psi.$$

This means that the space dependent part of Ψ, $\Psi_E(\vec{x})$, has to satisfy the following equation (analagous to the Time Independent Schrödinger Equation from non-relativistic quantum mechanics.)

$$[\vec{\alpha}\cdot\vec{\pi} + m\beta + V]\,\Psi_E(\vec{x}) = E\Psi_E(\vec{x}).$$

Writing this on "block" matrix form, and using the Pauli representation of the matrices ($\vec{\alpha}$ and β), we get

$$\begin{pmatrix}V+m & \vec{\sigma}\cdot\vec{\pi}\\ \vec{\sigma}\cdot\vec{\pi} & V-m\end{pmatrix}\begin{pmatrix}\phi(\vec{x})\\ \eta(\vec{x})\end{pmatrix} = E\begin{pmatrix}\phi(\vec{x})\\ \eta(\vec{x})\end{pmatrix}.$$

We write out each of the two equations

$$(V+m)\phi(\vec{x}) + (\vec{\sigma}\cdot\vec{\pi})\eta(\vec{x}) = E\phi(\vec{x}). \tag{2}$$

$$(\vec{\sigma}\cdot\vec{\pi})\phi(\vec{x}) + (V-m)\eta(\vec{x}) = E\eta(\vec{x}). \tag{3}$$

Solving for η in eqn (3) we get

$$\eta(\vec{x}) = \frac{(\vec{\sigma}\cdot\vec{\pi})\phi(\vec{x})}{E+m-V} \approx \frac{(\vec{\sigma}\cdot\vec{\pi})\phi(\vec{x})}{2m}, \tag{4}$$

where we have set $E+m-V \approx 2m$ since we are in the non-relativistic limit.

Putting the expression for η into eqn (2) we get

$$(V+m)\phi(\vec{x}) + \frac{(\vec{\sigma}\cdot\vec{\pi})^2}{2m}\phi(\vec{x}) = E\phi(\vec{x}). \tag{5}$$

Using that $E = m + E_{NR}$ we get

$$\left[\frac{(\vec{\pi}\cdot\vec{\sigma})^2}{2m} + V\right]\phi(\vec{x}) = E_{NR}\,\phi(\vec{x}). \tag{6}$$

b)

To go further we use the following property of the Pauli-matrices

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})I + i(\vec{a}\times\vec{b})\vec{\sigma}.$$

We then get

$$(\vec{\pi}\cdot\vec{\sigma})^2 = (\vec{\pi})^2 I + i(\vec{\pi}\times\vec{\pi})\vec{\sigma}. \tag{7}$$

It would seem that the cross product $\vec{\pi}\times\vec{\pi}$ is just zero, but we need to remember that $\pi = \vec{p} - e\vec{A}(x)$ is an operator. Note also that in the position representation $\vec{A}(x)$ is just a function, while $\vec{p}$ is a derivative operator. Using $\vec{p} = -i\vec{\nabla}$ and showing how the operator acts on the field ϕ we get

$$\begin{aligned}
(\vec{\pi}\times\vec{\pi})\phi &= (-i\vec{\nabla} - e\vec{A}) \times (-i\vec{\nabla} - e\vec{A})\phi \\
&= (-\vec{\nabla}\times\vec{\nabla} + ie\vec{\nabla}\times\vec{A} + ie\vec{A}\times\vec{\nabla} + e^2\vec{A}\times\vec{A})\phi \\
&= (ie\vec{\nabla}\times\vec{A} + ie\vec{A}\times\vec{\nabla})\phi \\
&= ie\vec{B}\phi + ie(\vec{\nabla}\phi)\times\vec{A} + ie\vec{A}\times(\vec{\nabla}\phi) \\
&= ie\vec{B}\phi,
\end{aligned}$$

where we have used that $\vec{B} = \vec{\nabla}\times\vec{A}$.

We can now put everything together

$$\left[\frac{(\vec{\pi}\cdot\vec{\sigma})^2}{2m}+V\right]\phi(\vec{x})=\left[\frac{(\vec{\pi})^2}{2m}-\frac{e}{2m}\vec{\sigma}\cdot\vec{B}+V\right]\phi(\vec{x})=E_{NR}\,\phi(\vec{x}). \tag{8}$$

We recognize this as the (time-independent) Scrödinger equation for a charged spin 1/2 particle in an electromagnetic field. I would like to note that in non-relativistic quantum mechanics spin was something we had to postulate, however we see that in the non relativistic limit of the Dirac equation the spin degrees of freedom arise naturally. Another thing that we see is that the Dirac equation predicts a gyromagnetic ratio equal to 2 (before corrections from QED). This is also a value that can not be derived from non-relativistic quantum mechanics. We see that the spin is an inherently relativistic quantity, and altough we see its effects in non-relativistic theories we need special relativity to understand where these effects come from.

2

We are to show that the trace of the matrices $M \cdot N$, is given as a sum of five terms

$$\begin{aligned} Tr(M\cdot N) = aTr(M)\cdot Tr(N) + bTr(M\gamma_5)\cdot Tr(\gamma_5 N) \\ +cTr(M\gamma_\mu)\cdot Tr(\gamma^\mu N) + dTr(M\gamma_\mu\gamma_5)\cdot Tr(\gamma_5\gamma^\mu N) \\ +eTr(M\sigma_{\mu\nu})\cdot Tr(\sigma^{\mu\nu}N). \end{aligned} \tag{9}$$

We use the different antisymmetric combinations of the Dirac matrices as a basis, and write M and N as superpositions of these basis matrices. There are 16 basis matrices in total, of five types

Type	Number of this type
I	1
γ^μ	4
$\sigma^{\mu\nu}=\frac{i}{2}[\gamma^\mu,\gamma^\nu],(\mu<\nu)$	6
$\gamma_5\gamma^\mu$	4
γ_5	1

An important thing to note about these basis matrices is that the only one with non-zero trace is the identity matrix, with trace of 4. Products of the different basis matrices are also traceless unless it is a product of one basis matrix with itself, in which case the matrix product becomes the identity (up to a sign). We can say that the basis matrices are orthogonal under trace.

We introduce the following notation for the 16 basis matrices

$$\Theta^a,$$

we can expand a general matrix, R, in this basis

$$R = \sum_{a=1}^{16} r^a \Theta^a. \tag{10}$$

To find r^a we introduce a set of corresponding basis matrices, $\Theta_a = (\Theta^a)^{-1}$, with the property that $\Theta^a \Theta_a = I$ (for a given value of a). The new basis differs from the old one only up to a sign, so the matrices are still orthogonal under trace.

We can likewise expand the matrix, R, in the new basis

$$R = \sum_{a=1}^{16} r_a \Theta_a.$$

We can use this new basis to find the coefficients r^a.

Multiplying eqn (10) by Θ_b from the right we get

$$R\Theta_b = \sum_{a=1}^{16} r^a \Theta^a \Theta_b = r^b \Theta^b \Theta_b + \sum_{a \neq b} r^a \Theta^a \Theta_b.$$

Now we can take trace on both sides

$$Tr(R\Theta_b) = r^b Tr(\Theta^b \Theta_b),$$

and we get

$$r^a = \frac{1}{4} Tr(R\Theta_a). \tag{11}$$

In the same way we have

$$r_a = \frac{1}{4} Tr(R\Theta^a). \tag{12}$$

Now we need to find the right expressions for Θ_a. It is clear that $(\gamma^\mu)^{-1} = \gamma_\mu$ in the sense that for example $\gamma^2\gamma_2 = I$. We could try the same approach and guess that $(\gamma_5\gamma^\mu)^{-1} = \gamma_5\gamma_\mu$ but $\gamma_5\gamma^\mu\gamma_5\gamma_\mu = -4$, so we get an extra minus sign. We can take care of this sign by commuting γ_μ and γ_5 past eachother and we get

$$(\gamma_5\gamma^\mu)^{-1} = \gamma_\mu\gamma_5.$$

Since $\gamma_5\gamma_5 = I$ it is its own inverse.

We also have that $(\sigma^{\mu\nu})^{-1} = \sigma_{\mu\nu}$. Thus we now know the exact expressions for both the basises

$$\Theta^a = \{I, \gamma^\mu, \sigma^{\mu\nu}\ (\mu < \nu), \gamma_5\gamma^\mu, \gamma_5\}, \tag{13}$$

$$\Theta_a = \{I, \gamma_\mu, \sigma_{\mu\nu}\ (\mu < \nu), \gamma_\mu\gamma_5, \gamma_5\}. \tag{14}$$

Now we have all we need to calculate the trace of the product of two matrices N and M. We expand M in the Θ^μ basis, and we expand N in the Θ_μ basis

$$\begin{aligned}
Tr(M \cdot N) &= \sum_a m^a n_a Tr(\Theta^a\Theta_a) + \sum_{a\neq b} m^a n_b Tr(\Theta^a\Theta_b) \\
&= 4\sum_a \frac{1}{4}Tr(M\Theta_a)\frac{1}{4}Tr(N\Theta^a) \\
&= \frac{1}{4}Tr(M)\cdot Tr(N) + \frac{1}{4}Tr(M\gamma_5)\cdot Tr(\gamma_5 N) \\
&+ \frac{1}{4}Tr(M\gamma_\mu)\cdot Tr(\gamma^\mu N) + \frac{1}{4}Tr(M\gamma_\mu\gamma_5)\cdot Tr(\gamma_5\gamma^\mu N) \\
&+ \frac{1}{4}Tr(M\sigma_{\mu\nu})\cdot Tr(\sigma^{\mu\nu}N).
\end{aligned} \tag{15}$$

We see then that all the coefficients are equal to $\frac{1}{4}$.

If we had not restricted ourselves to $\mu < \nu$, we would get $e = \frac{1}{8}$, since each of the six independent basis matrices in $\sigma^{\mu\nu}$ would contribute twice. I chose the restriction $\mu < \nu$ to make it clear how many linearly independent basis matrices $\sigma^{\mu\nu}$ contains, and to fit all the different basis matrices into the same general framework.

3

a)

We are looking at the first order term in the scattering of a positron in a given Coulomb potential. We are calculating the amplitude

$$S^{(1)}_{fi} = \langle f|S^{(1)}|i\rangle, \tag{16}$$

where

$$S^{(1)} = -ie \int d^4x \mathcal{N}[\bar{\psi(x)}\gamma^\mu \psi(x)] A_\mu(x).$$

The Dirac-field, ψ, we write in the following way

$$\begin{aligned} \psi &= \\ \psi^+ &+ \psi^- = \\ \underbrace{\sum_{r,\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}V}} c_r(\vec{p}) u_r(\vec{p}) e^{-ipx}}_{\text{destroys particles}} &+ \underbrace{\sum_{r,\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}V}} d_r^\dagger(\vec{p}) v_r(\vec{p}) e^{ipx}}_{\text{creates anti-particles}}. \end{aligned} \tag{17}$$

The adjoint field, $\bar{\psi}$, is defined as $\psi^\dagger\gamma^0 = \bar{\psi}$. This gives us

$$\begin{aligned} \bar{\psi} &= \\ \bar{\psi}^+ &+ \bar{\psi}^- = \\ \underbrace{\sum_{r,\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}V}} d_r(\vec{p}) \bar{v}_r(\vec{p}) e^{-ipx}}_{\text{destroys anti-particles}} &+ \underbrace{\sum_{r,\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}V}} c_r^\dagger(\vec{p}) \bar{u}_r(\vec{p}) e^{ipx}}_{\text{creates particles}}. \end{aligned} \tag{18}$$

The spinors, $u_r(\vec{p})$ and $v_r(\vec{p})$ are given as

$$u_r(\vec{p}) = \sqrt{E_{\vec{p}} + m} \begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_{\vec{p}}+m}\chi_r \end{pmatrix}, \quad v_r(\vec{p}) = \sqrt{E_{\vec{p}} + m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E_{\vec{p}}+m}\chi_r \\ \chi_r \end{pmatrix}.$$

The initial and final states are states with one positron, but with different spin and (3-) momentum. We write

$$|i\rangle = |e^+(\vec{p}\sigma)\rangle,$$

$$|f\rangle = |e^+(\vec{p'}\sigma')\rangle.$$

Writing out the full amplitude we get

$$\begin{aligned} S_{fi}^{(1)} &= \langle f|S^{(1)}|i\rangle \\ &= -ie\langle e^+(\vec{p'}\sigma')| \left[\int d^4x \mathcal{N}[\bar{\psi(x)}\gamma^\mu\psi(x)]A_\mu(x)\right] |e^+(\vec{p}\sigma)\rangle. \end{aligned} \tag{19}$$

Let us first look at what the normal ordering will give (I write out spinor indices to avoid ambiguous notation)

$$\mathcal{N}[\psi(\bar{x})\gamma^{\mu}\psi(x)] = [\bar{\psi}^{+}_{\alpha}\gamma^{\mu}_{\alpha\beta}\psi^{+}_{\beta} - \psi^{-}_{\beta}\gamma^{\mu}_{\alpha\beta}\bar{\psi}^{+}_{\alpha} + \bar{\psi}^{-}_{\alpha}\gamma^{\mu}_{\alpha\beta}\psi^{+}_{\beta} + \bar{\psi}^{-}_{\alpha}\gamma^{\mu}_{\alpha\beta}\psi^{-}_{\beta}].$$

We see that only the second term here was affected by the normal ordering.

Lets look at each of the terms in order

$$\bar{\psi}^{+}_{\alpha}\gamma^{\mu}_{\alpha\beta}\psi^{+}_{\beta}$$

This term will annihilate a particle and then annihilate an anti-particle. This term will not contribute to our process since ψ^{+} will just annihilate the initial state (there are no particles in the initial state).

$$-\psi^{-}_{\beta}\gamma^{\mu}_{\alpha\beta}\bar{\psi}^{+}_{\alpha}$$

This term will annihilate an anti-particle and then create an anti-particle. This term will contribute to our process since the final state is also a anti-particle.

$$\bar{\psi}^{-}_{\alpha}\gamma^{\mu}_{\alpha\beta}\psi^{+}_{\beta}$$

This term will annihilate a particle and then create a particle.This term will not contribute to our process since it will just annihilate the initial state.

$$\bar{\psi}^{-}_{\alpha}\gamma^{\mu}_{\alpha\beta}\psi^{-}_{\beta}$$

This term will create a particle and then create an anti-particle. This term will not contribute to our process since the final state is not a 3-particle state.

In conclusion we see that only the second term contributes to our process. We get

$$S^{(1)}_{fi} = ie\langle e^{+}(\vec{p'}\sigma')| \left[\int d^4x \psi^{-}_{\beta}(x)\gamma^{\mu}_{\alpha\beta}\bar{\psi}^{+}_{\alpha}(x)A_{\mu}(x) \right] |e^{+}(\vec{p}\sigma)\rangle. \qquad (20)$$

Lets first see what we get by acting with $\bar{\psi}^{+}$ on the initial state

$$\begin{aligned}\bar{\psi}^{+}|e^{+}(\vec{p}\sigma)\rangle &= \sum_{r,\vec{k}} \frac{1}{\sqrt{2E_{\vec{k}}V}} d_r(\vec{k})\bar{v}_r(\vec{k})e^{-ikx}|e^{+}(\vec{p}\sigma)\rangle \\ &= \sum_{r,\vec{k}} \frac{1}{\sqrt{2E_{\vec{k}}V}} \bar{v}_r(\vec{k})e^{-ikx}\delta_{\vec{k},\vec{p}}\delta_{r\sigma}|0\rangle \\ &= \frac{1}{\sqrt{2E_{\vec{p}}V}} \bar{v}_\sigma(\vec{p})e^{-ipx}|0\rangle.\end{aligned}$$

We can also find the rest of the inner product

$$\begin{aligned}\langle e^{+}(\vec{p'}\sigma')|\psi^{-}|0\rangle &= \langle e^{+}(\vec{p'}\sigma')| \sum_{r,\vec{k}} \frac{1}{\sqrt{2E_{\vec{k}}V}} d_r^{\dagger}(\vec{k})v_r(\vec{k})e^{ikx}|0\rangle \\ &= \sum_{r,\vec{k}} \frac{1}{\sqrt{2E_{\vec{k}}V}} v_r(\vec{k})e^{ikx}\langle e^{+}(\vec{p'}\sigma')|e^{+}(\vec{k}r)\rangle \\ &= \frac{1}{\sqrt{2E_{\vec{p'}}V}} v_{\sigma'}(\vec{p'})e^{ip'x}.\end{aligned}$$

This gives us

$$S_{fi}^{(1)} = ie\frac{1}{2V\sqrt{E_{\vec{p'}}E_{\vec{p}}}}\left[\int d^4x\, v_{\sigma'}^{\beta}(\vec{p'})e^{ip'x}\gamma_{\alpha\beta}^{\mu}A_\mu(x)\bar{v}_\sigma^{\alpha}(\vec{p})e^{-ipx}\right] \tag{21}$$

$$= ie\frac{1}{2V\sqrt{E_{\vec{p'}}E_{\vec{p}}}}\left[\int d^4x\, \bar{v}_\sigma^{\alpha}(\vec{p})\gamma_{\alpha\beta}^{\mu}A_\mu(x)v_{\sigma'}^{\beta}(\vec{p'})e^{-i(p-p')x}\right]. \tag{22}$$

In the given vectorpotential, $A_0 = \frac{C}{|\vec{x}|}$, $A_i = 0$, we get the following

$$S_{fi}^{(1)} = ie\frac{\bar{v}_\sigma(\vec{p})\gamma^0 v_{\sigma'}(\vec{p'})}{2V\sqrt{E_{\vec{p'}}E_{\vec{p}}}}\int d^4x\,\frac{C}{|\vec{x}|}e^{-i(p-p')x}. \tag{23}$$

b)

Taking the integral over the time part of eqn (23) we get

$$S_{fi}^{(1)} = ie\frac{\bar{v}_\sigma(\vec{p})\gamma^0 v_{\sigma'}(\vec{p'})}{2V\sqrt{E_{\vec{p'}}E_{\vec{p}}}}\int d^3\vec{x}\,\frac{C}{|\vec{x}|}e^{-i(\vec{p}-\vec{p'})\vec{x}}2\pi\delta(E_p - E_{p'}). \tag{24}$$

Using the integral identity given in the problem text we get

$$S_{fi}^{(1)} = ie\frac{\bar{v}_\sigma(\vec{p})\gamma^0 v_{\sigma'}(\vec{p'})}{2VE_{\vec{p}}}\frac{8\pi^2 C}{|\vec{p}-\vec{p'}|^2}\delta(E_p - E_{p'}). \tag{25}$$

We see from the delta-function in energy that the energy of the positron is conserved.

c)

Lets look more closely at the spinor product in eqn (25)

$$\bar{v}_\sigma(\vec{p})\gamma^0 v_{\sigma'}(\vec{p'}) = v_\sigma^\dagger(\vec{p})v_{\sigma'}(\vec{p'}).$$

First we look at the non-relativistic limit and quantize the spin along the same fixed axis.

This gives us

$$\begin{aligned} v_\sigma^\dagger(\vec{p})v_{\sigma'}(\vec{p'}) &= (E_{\vec{p}}+m)\begin{pmatrix}\chi_\sigma^\dagger\frac{\vec{\sigma}\cdot\vec{p}}{E_{\vec{p}}+m} & \chi_\sigma^\dagger\end{pmatrix}\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p'}}{E_{\vec{p}}+m}\chi_{\sigma'} \\ \chi_{\sigma'}\end{pmatrix} \\ &= (E_{\vec{p}}+m)\left[\chi_\sigma^\dagger\frac{\vec{\sigma}\cdot\vec{p}}{E_{\vec{p}}+m}\frac{\vec{\sigma}\cdot\vec{p'}}{E_{\vec{p}}+m}\chi_{\sigma'} + \chi_\sigma^\dagger\chi_{\sigma'}\right]. \end{aligned} \tag{26}$$

But we have $|\vec{p}| \ll m$, so we can neglect the first term and we get

$$v_\sigma^\dagger(\vec{p})v_{\sigma'}(\vec{p'}) \approx (E_{\vec{p}}+m)\chi_\sigma^\dagger\chi_{\sigma'} = (E_{\vec{p}}+m)\delta_{\sigma\sigma'}. \tag{27}$$

We see that in the non-relativistic limit, the amplitude vanishes unless $\sigma = \sigma'$, this means that (in this limit) the spin of the positron is approximately conserved.

When we look at the ultra-relativistic limit (or equivalently the massless limit), we want to show that the helicity is conserved. We therefore quantize the spin of the incoming particle along its momentum, $\vec{p}$. Correspondingly we quantize the spin of the outgoing particle along its momentum, $\vec{p'}$. Since we have quantized the spin in the direction of the momentum, the spinors χ_λ satisfy the following condition

$$(\vec{\sigma}\cdot\hat{p})\chi_\lambda = \pm\chi_\lambda,$$

where $\hat{p}$ is the unit vector along $\vec{p}$, and $\pm$ means positive or negative helicity.

In this limit $E_{\vec{p}} + m \approx |p|$ and we get

$$
\begin{aligned}
v_\sigma^\dagger(\vec{p}) v_{\sigma'}(\vec{p'}) &\approx (E_{\vec{p}} + m) \left[\chi_\lambda^\dagger (\vec{\sigma} \cdot \hat{p})(\vec{\sigma} \cdot \hat{p'}) \chi_{\lambda'} + \chi_\lambda^\dagger \chi_{\lambda'} \right] \\
&= (E_{\vec{p}} + m) \left[\pm \chi_\lambda^\dagger \chi_{\lambda'} + \chi_\lambda^\dagger \chi_{\lambda'} \right],
\end{aligned} \tag{28}
$$

where $\pm$ in the last case is $+$ if the helicity eigenvalues are equal and $-$ if they are different.

We see that in the ultra-relativistic limit, the amplitude vanishes if the initial and final states have opposite eigenvalues of helicity. We do not however get a simple delta-function in the case where the eigenvalues are the same, this is because the spin is quantized in different directions. In general this amplitude is dependent on the angle between the incoming and outgoing momenta.