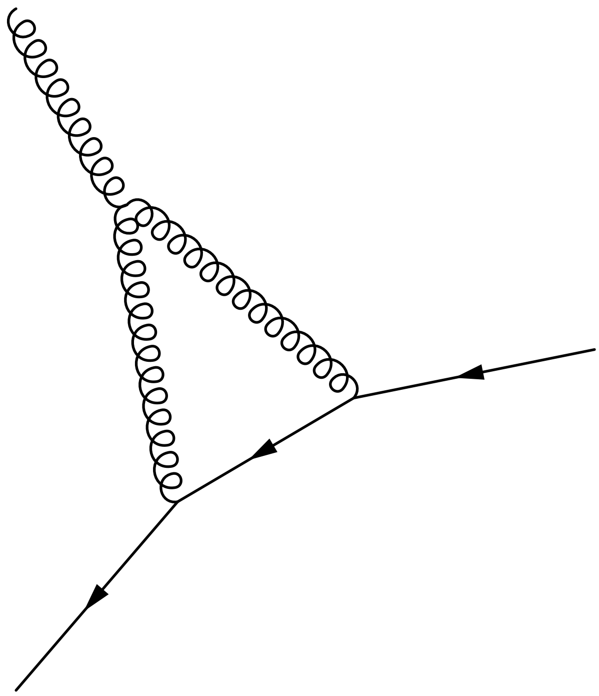
Mid-term exam, FYS-5120, 2014

Cand. nr.: 1

March 12, 2014



1 Problem 1

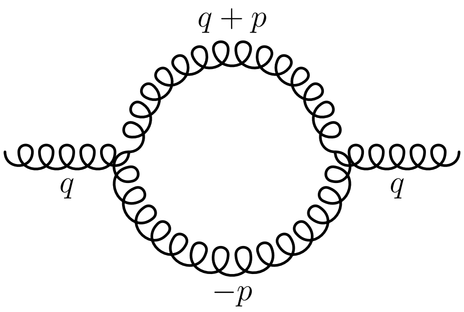


Figure 1: $\Pi_G^{\mu\nu}(q)^{ab}$. Gluon diagram calculated in Problem 1

We are calculating the gluonic vacuum polarization from this diagram. Where $\Pi_G^{\mu\nu}(q)^{ab}$ is given by (in $D = 4-\epsilon$ dimensions)

$$\Pi_G^{\mu\nu}(q)^{ab} = \frac{1}{2}\int d'p \frac{-i}{p^2}\frac{-i}{(p+q)^2} g^2 f^{acd} f^{bcd} N^{\mu\nu},$$

where

$$d'p = \frac{d^D p}{(2\pi)^D},$$

and

$$N^{\mu\nu} = [g^{\mu\rho}(q-p)^\sigma + g^{\rho\sigma}(2p+q)^\mu + g^{\sigma\mu}(-p-2q)^\rho] \\ \times \left[\delta_\rho^\nu (p-q)_\sigma + g_{\rho\sigma}(-2p+-q)^\nu + \delta_\sigma^\nu (p+2q)_\rho\right].$$

We can simplify the integrand by using Feynman parametrization

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[b+(a-b)x]^2}.$$

we get

$$\frac{1}{(p+q)^2}\frac{1}{p^2} = \int_0^1 \frac{dx}{[p^2+((p+q)^2-p^2)x]^2} = \int_0^1 \frac{dx}{[(p^2+2p\cdot qx+q^2x^2-q^2x^2+q^2x]^2} \\ = \int_0^1 \frac{dx}{[(p+qx)^2+q^2(x-x^2)]^2} = \int_0^1 \frac{dx}{[P^2-\Delta]^2},$$

where $P = p+qx$ and $\Delta = -x(1-x)q^2$.

Now, using $f^{acd}f^{bcd} = C_2(G)\delta^{ab} = 3\delta^{ab}$, we get the following integral

$$\Pi_G^{\mu\nu}(q)^{ab} = -\frac{3g^2}{2}\delta^{ab}\int_0^1 dx \int d'P \frac{1}{[P^2-\Delta]^2} N^{\mu\nu}.$$

Lets move on to the numerator, $N^{\mu\nu}$. First we just multiply out

$$N^{\mu\nu} = g^{\mu\nu}(q-p)\cdot(p-q) + (q-p)^\mu(-2p-q)^\nu + (p+2q)^\mu(q-p)^\nu +$$
$$(2p+q)^\mu(p-q)^\nu + D(2p+q)^\mu(-2p-q)^\nu + (2p+q)^\mu(p+2q)^\nu +$$
$$(p-q)^\mu(-p-2q)^\nu + (-p-2q)^\mu(-2p-q)^\nu + g^{\mu\nu}(-p-2q)\cdot(p+2q).$$

Now we want to write the numerator in terms of P, x, q and d. Note also that we can drop all terms linear in P since the denominator is even in P and these will not contribute to the integral. Using this, we get the following substitutions when changing variables

$$P^\mu P^\nu = g^{\mu\nu}\frac{P^2}{D},$$

$$p^2 = P^2 - 2P\cdot qx + q^2x^2 = P^2 + q^2x^2,$$

$$p^\mu p^\nu = P^\mu P^\nu - P^\mu q^\nu x - q^\mu P^\nu x + q^\mu q^\nu x^2 = g^{\mu\nu}\frac{P^2}{D} + q^\mu q^\nu x^2,$$

$$p^\mu q^\nu = P^\mu q^\nu - q^\mu q^\nu x = -q^\mu q^\nu x,$$

$$p\cdot q = -q^2 x.$$

Note that all the terms on the right hand side are symmetric in μ and ν so we can freely switch these indices when collecting terms. Lets put this into the numerator

$$\begin{aligned}
N^{\mu\nu} &= -g^{\mu\nu}[q^2 - 2p\cdot q + p^2 + 4q^2 + 4p\cdot q + p^2] - D[q^\mu q^\nu + 4p^\mu q^\nu + 4p^\mu p^\nu] \\
&+ 2[-2q^\mu p^\nu - q^\mu q^\nu + 2p^\mu p^\nu + p^\mu q^\nu + p^\mu q^\nu - p^\mu p^\nu + 2q^\mu q^\nu - 2q^\mu p^\nu \\
&+ 2p^\mu p^\nu + 4p^\mu q^\nu + q^\mu p^\nu + 2q^\mu q^\nu] \\
&= -g^{\mu\nu}[q^2(5 - 2x + 2x^2) + 2P^2] - D[q^\mu q^\nu(1 - 4x + 4x^2) + 4g^{\mu\nu}\frac{P^2}{D}] \\
&+ 6[g^{\mu\nu}\frac{P^2}{D} + q^\mu q^\nu x^2 - q^\mu q^\nu x + q^\mu q^\nu] \\
&= -g^{\mu\nu}P^2 6[1 - 1/D] - g^{\mu\nu}q^2[(2-x)^2 + (1+x)^2] \\
&+ q^\mu q^\nu[(2-D)(1-2x)^2 + 2(1+x)(2-x)].
\end{aligned}$$

We then get (using the table of Minkowski space integrals on page 807 is PS)

$$\Pi_G^{\mu\nu}(q)^{ab} = -\frac{3g^2}{2}\delta^{ab}\int_0^1 dx \int d'P \frac{1}{[P^2-\Delta]^2}[P^2 N_1^{\mu\nu} + N_2^{\mu\nu}] =$$

$$-\frac{3g^2}{2}\delta^{ab}\int_0^1 dx \left[N_1^{\mu\nu}\left(\frac{-i\Gamma(1-D/2)}{(4\pi)^{D/2}}\frac{D}{2}\left(\frac{1}{\Delta}\right)^{1-D/2}\right) + N_2^{\mu\nu}\left(\frac{i\Gamma(2-D/2)}{(4\pi)^{D/2}}\left(\frac{1}{\Delta}\right)^{2-D/2}\right)\right].$$

Where we define

$$N_1^{\mu\nu} = -g^{\mu\nu}6[1-1/D],$$

$$N_2^{\mu\nu} = -g^{\mu\nu}q^2[(2-x)^2+(1+x)^2] + q^\mu q^\nu[(2-D)(1-2x)^2 + 2(1+x)(2-x)].$$

Now we multiply $N_1^{\mu\nu}$ by Δ in both the numerator and the denominator. This gives us

$$\Pi_G^{\mu\nu}(q)^{ab} = \frac{3ig^2}{2(4\pi)^{D/2}}\delta^{ab}\int_0^1 dx \frac{1}{\Delta^{2-D/2}}\Big[\Gamma(1-D/2)g^{\mu\nu}q^2 3[D-1]x(1-x)$$

$$+\Gamma(2-D/2)g^{\mu\nu}q^2[(2-x)^2+(1+x)^2]$$

$$-\Gamma(2-D/2)q^\mu q^\nu[(2-D)(1-2x)^2+2(1+x)(2-x)]\Big].$$

Note that this does not have the transverse Lorentz structure required by the Ward identity. We have to combine it with the other terms to see what we get.

2 Problem 2

2.1 Tadpole diagram, $\Pi_{tp}^{\mu\nu}(q)^{ab}$

The tadpole diagram gives the following vacuum tensor contribution

$$\Pi_{tp}^{\mu\nu}(q)^{ab} = \frac{1}{2}\int d'p \frac{-i}{p^2}(-ig^2)N^{\mu\nu,ab},$$

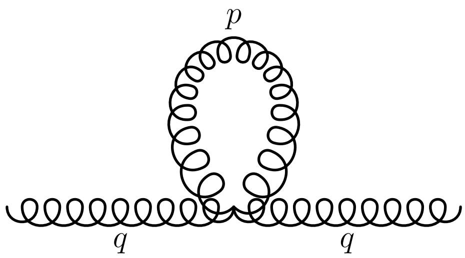


Figure 2: $\Pi^{\mu\nu}_{tp}(q)^{ab}$. Tadpole diagram calculated in Problem 2

where we have defined

$$N^{\mu\nu,ab} = g_{\rho\sigma}\delta^{cd}\Big(f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})$$

$$+f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho})$$

$$+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\Big).$$

Applying the deltafunction and metric tensor we get

$$N^{\mu\nu,ab} = 3\delta^{ab}(g^{\mu\nu}D - g^{\mu\nu})$$

$$+3\delta^{ab}(g^{\mu\nu}D - g^{\mu\nu})$$

$$= 6\delta^{ab}g^{\mu\nu}(D-1).$$

Thus the integral now becomes

$$\Pi^{\mu\nu}_{tp}(q)^{ab} = -3g^2\delta^{ab}g^{\mu\nu}(D-1)\int d'p\frac{1}{p^2} = 0,\ (D\to 4)$$

This integral gives zero in dimensional regularization as we go to four dimensions, put there is a pole at $D = 2$. We also note that one of the terms in the amplitude from problem one has the same pole (the term with $\Gamma(1 - D/2)$). If we write all the amplitudes on the same form we can see if we cancel the pole at $D = 2$. If we multiply our expression with $(q+p)^2$ both in the numerator and denominator we can use exactly the same Feynman parametrization as in problem 1.

$$\Pi^{\mu\nu}_{tp}(q)^{ab} = -3g^2\delta^{ab}g^{\mu\nu}(D-1)\int d'p\frac{1}{p^2}\frac{(q+p)^2}{(q+p)^2}$$

$$\Pi_{tp}^{\mu\nu}(q)^{ab} = -3g^2\delta^{ab}g^{\mu\nu}(D-1)\int_0^1 dx \int d'P \frac{1}{[P^2-\Delta]^2}N,$$

where

$$N = (p+q)^2 = P^2 + q^2(1-x)^2.$$

We apply the integral formulas and get

$$\Pi_{tp}^{\mu\nu}(q)^{ab} = -3g^2\delta^{ab}g^{\mu\nu}(D-1)\int_0^1 dx \Bigg[\left(\frac{-i\Gamma(1-D/2)}{(4\pi)^{D/2}}\frac{D}{2}\left(\frac{1}{\Delta}\right)^{1-D/2} \right)$$

$$+q^2(1-x)^2\left(\frac{i\Gamma(2-D/2)}{(4\pi)^{D/2}}\left(\frac{1}{\Delta}\right)^{2-D/2}\right)\Bigg]$$

$$= \frac{3ig^2}{2(4\pi)^{D/2}}\delta^{ab}\int_0^1 dx \frac{1}{\Delta^{2-D/2}}\Big[-\Gamma(1-D/2)g^{\mu\nu}q^2D(D-1)x(1-x)$$

$$-\Gamma(2-D/2)g^{\mu\nu}q^2 2(D-1)(1-x)^2\Big].$$

2.2 Ghost diagram, $\Pi_{gh}^{\mu\nu}(q)^{ab}$

Lets go on to the ghost diagram.

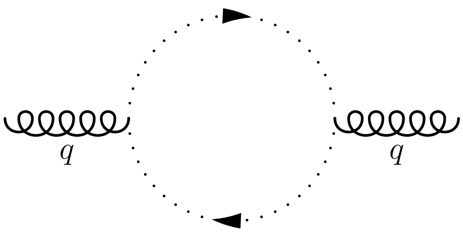


Figure 3: $\Pi_{gh}^{\mu\nu}(q)^{ab}$. Ghost diagram calculated in Problem 2.

The amplitude of the ghost diagram is given by

$$\Pi_{gh}^{\mu\nu}(q)^{ab} = -\int d'p \frac{i}{p^2}\frac{i}{(p+q)^2}(g^2)N^{\mu\nu,ab},$$

where

$$N^{\mu\nu,ab} = f^{dac}(p+q)^\mu f^{cbd}p^\nu = -3\delta^{ab}(p+q)^\mu p^\nu.$$

We get

$$\Pi^{\mu\nu}_{gh}(q)^{ab} = -3g^2\delta^{ab}\int d'p\frac{1}{p^2}\frac{1}{(p+q)^2}(p+q)^\mu p^\nu.$$

Here also we can use the same Feynman parametrization as in problem 1, we get

$$\Pi^{\mu\nu}_{gh}(q)^{ab} = -3g^2\delta^{ab}\int_0^1 dx\int d'P\frac{1}{[P^2-\Delta]^2}N^{\mu\nu},$$

where the numerator is given by

$$N^{\mu\nu} = (p+q)^\mu p^\nu = g^{\mu\nu}\frac{P^2}{D} + q^\mu q^\nu(x^2-x).$$

Using the integral formulas for dimensional regularization we get

$$\Pi^{\mu\nu}_{gh}(q)^{ab} = -3g^2\delta^{ab}\int_0^1 dx\left[\frac{g^{\mu\nu}}{D}\left(\frac{-i\Gamma(1-D/2)}{(4\pi)^{D/2}}\frac{D}{2}\left(\frac{1}{\Delta}\right)^{1-D/2}\right)\right.$$
$$\left.+q^\mu q^\nu(x^2-x)\left(\frac{i\Gamma(2-D/2)}{(4\pi)^{D/2}}\left(\frac{1}{\Delta}\right)^{2-D/2}\right)\right].$$

Lets write this on the same form as in the others so we can combine all the expressions

$$\Pi^{\mu\nu}_{gh}(q)^{ab} = \frac{3ig^2}{2(4\pi)^{D/2}}\delta^{ab}\int_0^1 dx\frac{1}{\Delta^{2-D/2}}\Big[-\Gamma(1-D/2)g^{\mu\nu}q^2x(1-x)$$
$$+\Gamma(2-D/2)q^\mu q^\nu 2x(1-x)\Big].$$

2.3 Combining all three diagrams

Now we can combine the three contributions to see if they combine to an expression on the required form.

$$\Pi^{\mu\nu}(q)^{ab} = \Pi^{\mu\nu}_G(q)^{ab} + \Pi^{\mu\nu}_{tp}(q)^{ab} + \Pi^{\mu\nu}_{gh}(q)^{ab}$$
$$= \frac{3ig^2}{2(4\pi)^{D/2}}\delta^{ab}\int_0^1 dx\frac{1}{\Delta^{2-D/2}}\Big[\Gamma(1-D/2)g^{\mu\nu}q^2\Big(x(1-x)(3[D-1]-D(D-1)-1)\Big)$$
$$+\Gamma(2-D/2)g^{\mu\nu}q^2\Big((2-x)^2+(1+x)^2-2(D-1)(1-x)^2\Big)$$

$$+\Gamma(2-D/2)q^\mu q^\nu\Big(-(2-D)(1-2x)^2-2(1+x)(2-x)+2x(1-x)\Big)\Big].$$

We can combine the two first terms by using

$$(1-D/2)\Gamma(1-D/2)=\Gamma(2-D/2).$$

If we are to get the correct form of the amplitude we need the coefficients of $g^{\mu\nu}q^2$ to be equal to minus the coefficient of $q^\mu q^\nu$ this means that the following two integrals over x need to be equal.

$$\int_0^1 dx\frac{1}{\Delta^{2-D/2}}\Big[(2-x)^2+(1+x)^2-2(D-1)(1-x)^2+2(D-2)x(1-x)\Big]$$

$$=\int_0^1 dx\frac{1}{\Delta^{2-D/2}}\Big[(2-D)(1-2x)^2+2(1+x)(2-x)-2x(1-x)\Big].$$

It is not entirely obvious, but we can show that these two integrals are equal if we exploit the symmetry of the denominator under the change of variables $x\to(1-x)$. In this way we can set

$$x=\frac{1}{2}(x+x)=\frac{1}{2}(x+(1-x))=\frac{1}{2}$$

If we do this the expressions both get the form

$$\int_0^1 dx\frac{1}{\Delta^{2-D/2}}\Big[2+D+4(2-D)x^2\Big].$$

Thus we get

$$\Pi^{\mu\nu}(q)^{ab}=\frac{3ig^2}{2(4\pi)^{D/2}}\delta^{ab}\int_0^1 dx\frac{\Gamma(2-D/2)}{\Delta^{2-D/2}}(g^{\mu\nu}q^2-q^\mu q^\nu)\Big[D+2+4(2-D)x^2\Big].$$

We can evaluate the integral over x using the formula given in the problem text

$$\int_0^1 dx\frac{D+2+4(2-D)x^2}{\Delta^{2-D/2}}=(-q^2)^{D/2-2}\Big[(D+2)\frac{\Gamma(D/2-1)\Gamma(D/2-1)}{\Gamma(D-2)}$$

$$+4(2-D)\frac{\Gamma(D/2+1)\Gamma(D/2-1)}{\Gamma(D)}\Big].$$

This gives us the final amplitude

$$\Pi^{\mu\nu}(q)^{ab}=\frac{3ig^2}{2(4\pi)^{D/2}}\delta^{ab}\frac{\Gamma(2-D/2)}{\Delta^{2-D/2}}(g^{\mu\nu}q^2-q^\mu q^\nu)(-q^2)^{D/2-2}$$

$$\times\left[(D+2)\frac{\Gamma(D/2-1)\Gamma(D/2-1)}{\Gamma(D-2)}\right.$$
$$\left.+4(2-D)\frac{\Gamma(D/2+1)\Gamma(D/2-1)}{\Gamma(D)}\right].$$

This is the whole amplitude (to all orders in ϵ). It can be simplified a bit by using the recursion relations of the gamma functions, but we would not learn much. It is however interesting to look at the divergent part of the expression.

The only part of this amplitude that diverges as $D \to 4$ is the gamma function, $\Gamma(2-D/2)$.

If we let $D \to 4$ in the finite parts we get the divergent part of the amplitude (technically only the first term in the expansion of $\Gamma(2-D/2)$ is divergent)

$$\Pi^{\mu\nu}(q)^{ab}_{DIV} = \frac{3ig^2}{(4\pi)^2}\delta^{ab}\frac{5}{3}\Gamma(2-D/2)(g^{\mu\nu}q^2 - q^\mu q^\nu).$$

3 Problem 3

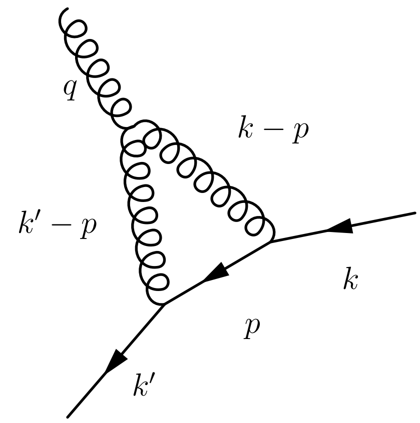


Figure 4: $\Delta\Gamma^{\mu,a}(k,k',q)$. Vertex correction term calculated in problem 3

The amplitude coming from the vertex correction with three gluons (not the three gluon vertex!). In the massless fermion and zero external gluon momentum limit, the amplitude is given by

$$\Delta\Gamma^{\mu,a}(k,k) = \int d'p(ig\gamma_\nu t^b)\frac{i\not{p}}{p^2}(ig\gamma_\rho t^c)\frac{(-i)^2}{((k-p)^2)^2}$$
$$\times g f^{abc}[g^{\mu\nu}(k-p)^\rho - 2g^{\nu\rho}(k-p)^\mu + g^{\rho\mu}(k-p)^\nu].$$

We can simplify this by using the relation (p 806 PS)

$$f^{abc}t^b t^a = \frac{1}{2}iC_2(G)t^a$$

we get

$$\Delta\Gamma^{\mu,a}(k,k) = \frac{-3g^3}{2}t^a\int d'p\gamma_\nu\not{p}\gamma_\rho\frac{N^{\mu\nu\rho}}{p^2((k-p)^2)^2},$$

where

$$N^{\mu\nu\rho} = g^{\mu\nu}(k-p)^\rho - 2g^{\nu\rho}(k-p)^\mu + g^{\rho\mu}(k-p)^\nu.$$

Now, applying Feynman parametrization we get

$$\frac{1}{((k-p)^2)^2p^2} = 2\int_0^1 \frac{xdx}{\left[(k-p)^2x + p^2(1-x)\right]^3}.$$

We change the variables to $P = p - kx$ and this becomes

$$2\int_0^1 \frac{xdx}{\left[(k-p)^2x+p^2(1-x)\right]^3}$$
$$= 2\int_0^1 \frac{xdx}{\left[P^2-\Delta\right]^3},$$

where $\Delta = -k^2x(1-x)$.

Lets work on the numerator. First we note that we have the following transformations in the numerator of this integral

$$p^\mu p^\nu = (P+kx)^\mu(P+kx)^\nu = P^\mu P^\nu + P^\mu k^\nu x + k^\mu x P^\nu + k^\mu k^\nu x^2 = \frac{P^2 g^{\mu\nu}}{D} + k^\mu k^\nu x^2,$$

$$p^\mu k^\nu = k^\mu k^\nu x.$$

We get

$$\gamma_\nu\gamma_\sigma\gamma_\rho p^\sigma[g^{\mu\nu}(k-p)^\rho - 2g^{\nu\rho}(k-p)^\mu + g^{\rho\mu}(k-p)^\nu]$$
$$= \gamma_\nu\gamma_\sigma\gamma_\rho\Big[g^{\mu\nu}(k^\sigma k^\rho x(1-x) - \frac{P^2}{D}g^{\sigma\rho}) - 2g^{\nu\rho}(k^\sigma k^\mu x(1-x) - \frac{P^2}{D}g^{\sigma\mu})$$
$$+ g^{\rho\mu}(k^\sigma k^\nu x(1-x) - \frac{P^2}{D}g^{\sigma\nu})\Big]$$
$$= \frac{P^2}{D}\Big(2\gamma^\rho\gamma^\mu\gamma_\rho - \gamma^\mu\gamma^\rho\gamma_\rho - \gamma^\sigma\gamma_\sigma\gamma^\mu\Big)$$
$$+2x(1-x)(\gamma^\mu \not{k}\not{k} - \gamma^\rho\not{k}\gamma_\rho k^\mu)$$
$$= -\frac{4(D-1)P^2}{D}\gamma^\mu + 2x(1-x)(\gamma^\mu k^2 + (D-2)\not{k}k^\mu).$$

Now we are ready to do the integral over P, we get

$$\Delta\Gamma^{\mu,a}(k,k) = \frac{3g^3}{2}t^a\int_0^1 dx 2x\left[\frac{4(D-1)i\gamma^\mu}{D(4\pi)^{D/2}}\frac{D}{2}\frac{\Gamma(2-D/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{2-D/2}\right.$$
$$\left. + 2x(1-x)(\gamma^\mu k^2 + (D-2)\not{k}k^\mu)\frac{i\Gamma(3-D/2)}{(4\pi)^{D/2}\Gamma(3)}\left(\frac{1}{\Delta}\right)^{3-D/2}\right]$$

$$= \frac{3ig^3}{(4\pi)^{D/2}}t^a \underbrace{\int_0^1 dx\frac{x}{\Delta^{2-D/2}}}_{\text{Finite}}\left[\underbrace{\frac{4(D-1)}{4}\gamma^\mu\Gamma(2-D/2)}_{\text{Divergent}} - \underbrace{\frac{\gamma^\mu k^2 + (D-2)\not{k}k^\mu}{k^2}\Gamma(3-D/2)}_{\text{Finite}}\right].$$

We can use the function given in the exercise text to do the integral over x

$$\int_0^1 dx \frac{x}{\Delta^{2-D/2}} = (-k^2)^{D/2-2} \frac{\Gamma(D/2)\Gamma(D/2-1)}{\Gamma(D-1)}.$$

We get

$$\Delta\Gamma^{\mu,a}(k,k) = \frac{3ig^3}{(4\pi)^{D/2}} t^a (-k^2)^{D/2-2} \frac{\Gamma(D/2)\Gamma(D/2-1)}{\Gamma(D-1)} \left[\frac{4(D-1)}{4} \gamma^\mu \Gamma(2-D/2) \right.$$
$$\left. - \frac{\gamma^\mu k^2 + (D-2) \not{k} k^\mu}{k^2} \Gamma(3-D/2) \right].$$

If we want to look at just the divergent part of this then we get

$$\Delta\Gamma^{\mu,a}_{DIV}(k,k) = \frac{3ig^3}{(4\pi)^2} t^a \gamma^\mu \left[\frac{3}{2} \Gamma(2-D/2) \right].$$

4 Problem 4

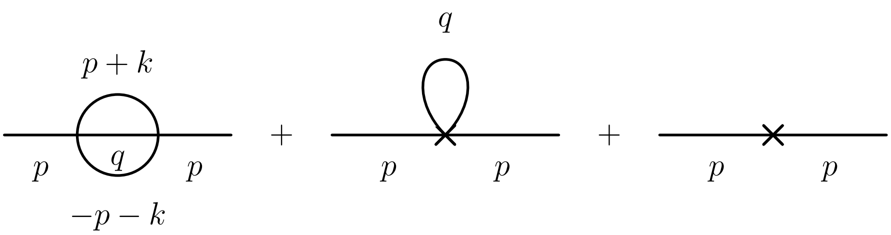


Figure 5: Diagrams A, B and C. 2 - loop contributions to 2-point function (self energy) in ϕ^4 theory

For the first diagram we get the amplitude

$$A = \frac{(-i\lambda)^2}{3!}\int d'kd'q \frac{i}{(q+k)^2-m^2}\frac{i}{q^2-m^2}\frac{i}{(p+k)^2-m^2}$$
$$= \frac{-i\lambda^2}{3!}\int d'kd'q \frac{1}{m^2-(q+k)^2}\frac{1}{m^2-q^2}\frac{1}{m^2-(p+k)^2}$$

Now we preform a Wick rotation (i will not bother to re-label all variables with a subscript E, but this is to be understood implicitly until we go back to Minkowski space)

$$p^2 \to -p^2,$$
$$d'k \to id'k.$$

$$A_E = \frac{-i\lambda^2(i)^2}{3!}\int d'kd'q \frac{1}{m^2+(q+k)^2}\frac{1}{m^2+q^2}\frac{1}{m^2+(p+k)^2}$$
$$= \frac{i\lambda^2}{3!}\int d'kd'Q \int_0^1 dx \frac{1}{[Q^2+\Delta]^2}\frac{1}{m^2+(p+k)^2},$$

where $Q = q + xk$ and $\Delta = m^2 + x(1-x)k^2$. We get

$$A_E = \frac{i\lambda^2}{3!}\int d'k \int_0^1 dx \frac{1}{(4\pi)^{D/2}}\frac{\Gamma(2-D/2)}{[m^2+x(1-x)k^2]^{2-D/2}}\frac{1}{m^2+(p+k)^2}$$

$$= \frac{i\lambda^2}{3!}\int d'k \int_0^1 dx[x(1-x)]^{[D/2-2]}\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}}\left[\frac{1}{\frac{m^2}{x(1-x)}+k^2}\right]^{2-D/2}\frac{1}{m^2+(p+k)^2}$$

Now, using the general Feynman parametrization formula (p.342 PS) we get

$$= \frac{i\lambda^2}{3!} \int d'k \int_0^1 dxdy [x(1-x)]^{D/2-2} \frac{\Gamma(2-D/2)}{(4\pi)^{D/2}} \frac{\Gamma(3-D/2)}{\Gamma(2-D/2)\Gamma(1)}$$

$$y^{1-D/2} \left[\frac{1}{y\frac{m^2}{x(1-x)} + yk^2 + (1-y)[m^2 + (p+k)^2]} \right]^{3-D/2}$$

$$= \frac{i\lambda^2}{3!} \int d'k \int_0^1 dxdy [x(1-x)]^{D/2-2} y^{1-D/2} \frac{\Gamma(3-D/2)}{(4\pi)^{D/2}} \left[\frac{1}{K^2 + \Delta} \right]^{3-D/2},$$

where $K = k + (1-y)p$ and $\Delta = p^2 y(1-y) + m^2(1-y) + y\frac{m^2}{x(1-x)}$.

$$A_E = \frac{i\lambda^2}{3!} \int_0^1 dxdy [x(1-x)]^{D/2-2} y^{1-D/2} \frac{\Gamma(3-D)}{(4\pi)^D}$$

$$\times \left[\frac{1}{p^2 y(1-y) + m^2(1-y) + y\frac{m^2}{x(1-x)}} \right]^{3-D}.$$

Now we can take the limit $m \to 0$ we get the following

$$A_E = \frac{i\lambda^2}{3!(4\pi)^D} \int_0^1 dxdy [x(1-x)]^{D/2-2} y^{1-D/2} \frac{\Gamma(3-D)}{[p^2 y(1-y)]^{3-D}}.$$

Now we also take the limit $D = 4 - \epsilon \to 4$, using the limits

$$\lim_{\epsilon \to 0} \int_0^1 dx [x(1-x)]^{-\epsilon/2} = 1,$$

$$\lim_{\epsilon \to 0} \int_0^1 dy y^{-\epsilon/2} (1-y)^{1-\epsilon} = \frac{1}{2},$$

$$\epsilon \to 0 : x^{1-\epsilon} \approx x - x\ln(x)\epsilon + ...$$

$$\epsilon \to 0 : \Gamma(\epsilon - 1) \approx -\frac{1}{\epsilon} + ...$$

We now get

$$A_E = \frac{i\lambda^2}{12(4\pi)^4} \Gamma(\epsilon - 1)(p^2)^{1-\epsilon} + ... = -ip^2 \frac{\lambda^2}{12(4\pi)^4} \left(\frac{1}{\epsilon} - \ln(p^2) + ... \right).$$

Going back to Minkowski-space we get

$$A = -ip^2 \frac{\lambda^2}{12(4\pi)^4} \left(-\frac{1}{\epsilon} + \ln(-p^2) + ... \right).$$

It seems I have a slightly different result compared to that found in the exercise text in PS (the sign of the argument of the loartithm). I am not sure if it is because of an error in PS or an error in my calculation.

The second (tadpole) diagram, B, is given by the following amplitude

$$B = \frac{-i\delta_\lambda}{2} \int d'q \frac{i}{q^2 - m^2} = \frac{-i\delta_\lambda}{2(4\pi)^{D/2}} \frac{\Gamma(1 - D/2)}{(m^2)^{1-D/2}} \propto m^2 = 0, \ (m \to 0, D \to 4).$$

We see that this diagram vanishes in the massless limit as $D \to 4$. This means that the last diagram, C, must cancel the divergence we got from A.