Problem set 1 FYS9130 – Week 35

1. **Variation of action**

   An action integral is given by

   $$S = \int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}R + \mathcal{L}_m\right) \tag{1}$$

   where $\mathcal{L}_m = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}F_{\mu\nu}$ and $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}$ depends only on the field A_μ and not the metric $g_{\mu\nu}$.

   a) Show that when variating the action with respect to the metric $g_{\mu\nu}$ the resulting equations of motion are

   $$M_P^2 E_{\mu\nu} = F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{2}$$

   b) Show that when variating the action with respect to the field A_μ the resulting equations of motion are the source-free Maxwell equations

   $$F^{\mu\nu}{}_{;\nu} = 0 \tag{3}$$

2. **Variation of Riemann tensor**

   Show that

   $$\delta R^\alpha{}_{\mu\beta\nu} = \delta\Gamma^\alpha{}_{\mu\nu;\beta} - \delta\Gamma^\alpha{}_{\mu\beta;\nu} \tag{4}$$

3. **Contracted Bianchi identities**

   Use the Bianchi identity

   $$R_{\mu\nu[\rho\sigma;\tau]} = 0 \tag{5}$$

   and that for a symmetric tensor $S_{\mu\nu}$ we have

   $$S_{\mu\nu;\alpha\beta} - S_{\mu\nu;\beta\alpha} = S_{\mu\sigma}R^\sigma{}_{\nu\alpha\beta} + S_{\sigma\nu}R^\sigma{}_{\mu\alpha\beta} \tag{6}$$

   to find

   a) $R^{\mu\nu\rho\sigma}{}_{;\mu} = R^{\nu\sigma;\rho} - R^{\nu\rho;\sigma}$

   b) $R^{\mu\nu}{}_{;\nu} = \frac{1}{2}R^{;\mu}$

   c) $R^{\alpha\nu;\beta}{}_\nu = \frac{1}{2}R^{;\alpha\beta} + R^{\alpha\sigma}R^\beta{}_\sigma - R_{\mu\nu}R^{\alpha\mu\beta\nu}$

   d) $R^{\mu\alpha\nu\beta}{}_{;\mu\nu} = \Box R^{\alpha\beta} - \frac{1}{2}R^{;\alpha\beta} - R^{\alpha\sigma}R^\beta{}_\sigma + R_{\mu\nu}R^{\alpha\mu\beta\nu}$

   e) $R^{\mu\nu}{}_{;\mu\nu} = \frac{1}{2}\Box R$

4. **Extra problem**

   Prove equation (6).