Problem set 7 FYS9130 – Week 42

1. **Week field limit**

   Assume that we can write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is a small pertubation, $|h_{\mu\nu}| << 1$. Show that to first order in $h_{\mu\nu}$ we have

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{1}$$

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}\left(h_{\beta\mu,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu,\beta}\right) \tag{2}$$

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\mu\rho,\nu\sigma} - h_{\nu\sigma,\mu\rho}\right) \tag{3}$$

$$R_{\mu\nu} = \frac{1}{2}\left(h^{\sigma}{}_{\mu,\nu\sigma} + h^{\sigma}{}_{\nu,\mu\sigma} - h_{,\mu\nu} - \Box h_{\mu\nu}\right) \tag{4}$$

$$R = h^{\mu\nu}{}_{,\mu\nu} - \Box h \tag{5}$$

2. **Lagrangian**

   Show that by varying the action $S = \int d^n x \mathcal{L}$ given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(h^{\mu\nu}{}_{,\mu}h_{,\nu} - h^{\mu\rho,\sigma}h_{\mu\sigma,\rho} + \frac{1}{2}h^{\rho\sigma,\mu}h_{\rho\sigma,\mu} - \frac{1}{2}h^{,\mu}h_{,\mu}\right) \tag{6}$$

   with respect to $h_{\mu\nu}$, we get the first-order vacuum Einstein equations.

3. **Gauge freedom**

   Show that an transformation $h_{\mu\nu} \to h_{\mu\nu} + X_{\mu,\nu} + X_{\nu,\mu}$ leaves the first-order Riemann tensor unchanged for any vector X_{μ}.